Volume 41 *July 1955* *Number 7*

Proceedings of the
National Academy of Sciences
of the United States of America

The Proceedings of the National Academy of Sciences

OFFICERS OF THE ACADEMY

DETLEV W. BRONK
President

GEORGE W. CORNER
Vice President

JOHN G. KIRKWOOD
Foreign Secretary

HUGH L. DRYDEN
Home Secretary

WILLIAM J. ROBBINS
Treasurer

EDITORIAL BOARD OF THE PROCEEDINGS

WENDELL M. STANLEY
Chairman

HUGH L. DRYDEN, *Home Secretary* JOHN G. KIRKWOOD, *Foreign Secretary*
EDWIN B. WILSON, *Managing Editor*

J. W. BEAMS	G. W. BEADLE	GREGORY BREIT
R. E. CLELAND	E. A. DOISY	L. C. DUNN
W. W. RUBEY	G. C. EVANS	W. F. LIBBY
HARLOW SHAPLEY	C. H. GRAHAM	J. VON NEUMANN
F. E. TERMAN	J. T. PATTERSON	PAUL WEISS

Subscription price is $12.50 per volume (12 issues); the price of a single issue is $1.25. Make all remittances payable to The University of Chicago Press in U.S. currency or its equivalent. Authorized agent for the British Commonwealth, except North America and Australasia: Cambridge University Press, Bentley House, 200 Euston Road, London, N. W. 1, England.

Editorial correspondence should be addressed to the PROCEEDINGS of the NATIONAL ACADEMY OF SCIENCES, Attention Miss Mary D. Alexander, University of Chicago Press, 5750 Ellis Avenue, Chicago 37, Illinois.

Business correspondence should be addressed to the Journals Division, University of Chicago Press, 5750 Ellis Avenue, Chicago 37, Illinois.

Subscribers are requested to notify The University of Chicago Press and their local postmaster immediately of change of address.

Microfilms of complete volumes of this journal are available to regular subscribers only and may be obtained at the end of the volume year from University Microfilms, 313 N. First Street, Ann Arbor, Michigan.

Entered as second-class matter June 21, 1920 at the Post-office at Easton, Pennsylvania, under the Act of August 24, 1912. Accepted for mailing at a special rate of postage provided for in Section 1103, Act of October 3, 1917. Authorized June 18, 1920.

PRINTED IN THE U. S. A.

THE PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES
is published for
THE NATIONAL ACADEMY OF SCIENCES
by
THE UNIVERSITY OF CHICAGO PRESS

Proceedings of the
NATIONAL ACADEMY OF SCIENCES

Volume 41 · Number 7 · July 15, 1955

*SOME CRITICISMS OF "ON THE ORIGIN OF THE LUNAR SURFACE FEATURES" BY G. P. KUIPER**

By Harold C. Urey

INSTITUTE FOR NUCLEAR STUDIES, UNIVERSITY OF CHICAGO

Communicated May 12, 1955

In a recent issue of these Proceedings, Professor Kuiper[1] has presented some views in regard to the structure and history of the moon which are in sharp disagreement with views presented by the writer[2, 3] and others[4, 5] on these matters. It is the purpose of this note to answer Kuiper's argument.

Kuiper has made additional observations on the moon with the 82-inch reflector at McDonald Observatory; but, so far as I can deduce from his discussion, he has not observed anything markedly different from what has been previously observed and recorded in the extensive literature on the moon. It may be that he has observed some new details, but the information bearing on his theories has been available before. His discussion, which takes issue with my ideas, really derives from his view that the moon was completely melted in its early history because of radioactive heating. In the fall of 1953 I remarked to Professor Kuiper in a discussion of the increased age of the solar system as presented by Patterson[6] that it seemed to me that the constants, i.e., the age of the solar system and amounts of the radioactive elements, should be such that the moon would not have melted. From this remark it should have been obvious that, at that time, I had already made Dr. Kuiper's calculations in regard to the melting of the moon. In a recent paper,[7] I discussed the abundances of the radioactive isotopes relative to this question of a melted moon, as well as the heat balance of Mars and the earth. From this paper it is evident that serious difficulties in regard to the explanation of the properties of the earth, the moon, and Mars are encountered, unless the abundances of K, U, and Th are much less in the planets than they are in the chondritic meteorites. That such should be the case is not surprising in view of the quite obvious differentiation of the meteorites in other ways.

Kuiper maintains that the moon's irregular shape is not due to a frozen tidal wave but was produced by the irregular fall of planetesimals onto its surface and that this occurred while the moon was completely molten. Certainly, if the moon were molten, its surface would be an equipotential surface appropriate to its distance from the earth at that time. In this case the irregular shape must be due to a frozen tidal wave, in contradiction to Kuiper's other arguments.

He argues that the moon's elongated axis, now pointed toward and away from the earth, was once oriented along its orbital path. For this he uses dynamical

arguments which assume that the moon had a rigid structure as it moved from a position nearer the earth; but he also assumes that at the time when it was oriented with its newly acquired elongated axis in the direction of its path it was also completely melted and hence not rigid at all. Sir George Darwin has discussed the question of plastic deformation of the earth and has shown convincingly that dynamical problems of this kind are very difficult to handle rigorously. Kuiper's very brief discussion of this subject is at least internally inconsistent.

Kuiper postulates that the moon accumulated at low temperatures, an assumption which had previously been made explicitly. He then points out that radioactive heating would raise its temperature. If the amounts of radioactive substances in the moon are the same as those in the meteorites as reported by Patterson,[6] melting would occur in about 5×10^8 years or more[7] (the timing of events here is difficult to understand and will be discussed later). It seems to me that expansion would occur during this heating process both because of thermal expansion and because of expansion on melting, and hence cracks would occur in the moon's surface, and the liquid silicates would flow up through such cracks and flood the surface. All these effects are ignored in Kuiper's presentation. In fact, it is impossible to decide whether in his opinion the liquid lunar silicates are less or more dense than the solids. It seems most improbable that any surface features of the moon acquired previous to complete melting would remain after this melting process, as he assumes.

Next he postulates that solidification of the moon took place with the radioactive elements concentrating almost entirely at the surface. Now the solid silicates are definitely assumed to be more dense than their liquids, and hence solidification begins to occur in the moon's interior; this is compared with a similar process of solidification of the earth. In previous publications[2, 3] I showed that the moon would lose heat from its interior very slowly by conduction, and a revision of these calculations with more recent data confirms the previous conclusions. It is difficult to present briefly the serious objections to Kuiper's assumptions. Even in 4.5×10^9 years the center of the moon would lose very little heat through conduction and hence would remain near its melting point because of original heat alone, if at any time, for any reason, it had been completely melted. Even a small fraction of the postulated concentrations of radioactive elements, only 15 per cent of those reported for the chondritic meteorites, would maintain the interior of the moon out to 0.8 of its radius at the melting point of silicates or, indeed, cause extensive melting. It is difficult to believe that all the radioactive elements were removed by a solidification process. This certainly has not occurred in the case of the earth. If complete separation of the radioactive elements has occurred in the case of the earth, the mean concentration of potassium, uranium, and thorium in the earth as a whole must be very low, and very little heating due to radioactive substances would occur in matter of this concentration. It would then be necessary to assume much higher concentrations in the moon and the parent-bodies of the meteorites if Kuiper's postulated heating of these objects occurs. This necessitates some consideration of possible processes for making large astronomical objects of markedly different chemical composition, which Kuiper has not done. Even using much lower abundances for the radioactive elements in the earth,[7] it appears that only one-half to one-fourth of these elements are in the earth's crust. No consistent

explanation for the rigidity of the moon as indicated by its irregular shape has been offered by Kuiper, for the moon could hardly retain an irregular shape if its interior were melted. It seems to me that the moon accumulated at low temperatures and that the concentrations of radioactive elements are so low that the moon has never melted. I advanced this view some years ago, and there appears to be no other way to account for the facts.

Kuiper believes that a "sediment ring," consisting of solid objects of the kind that bombarded the moon early in its history, previously existed about the earth. He believes that this ring accumulated by molecular condensation of a "silicate gas," i.e., molecular weight of 50, at some 100° K. during some 10^2–10^3 years. In another publication[8] he accounted for the planetesimals, previously postulated by the writer to explain the chemical properties of the earth and meteorites, by a similar mechanism of accumulation during some 10^7 years. At the concentrations postulated, "silicate gas" or, in fact, water vapor, which he also discusses, would be highly supersaturated. Radioactive substances were present, and in their presence (or even in their absence) nucleation would occur in seconds, and no growth of large objects would occur. Kuiper says: "The absence of such bodies would be more puzzling than their presence." That such objects existed is certain, since objects did collide with the moon; but if physical-chemical theory and observation are valid, they could not have accumulated as he assumes. Also it should be noted that no such sediment rings exist now in the neighborhood of any planets except Saturn, and the existence of this ring seems to be due to the breakup of a satellite which formed or tried to form within the Roche limit of the planet.

The maria are markedly concentrated in the center of the moon's disk. Kuiper has turned the moon through 90° as it passed through his postulated "sediment ring" and then has the planetesimals hitting the advanced and trailing regions of the moon. But in most cases the moon would surely perturb the orbits of these planetesimals before collision occurred, and they would be scattered out of the ring into approximately elliptical orbits. Subsequently, collisions from all directions would occur, and hence this mechanism gives no explanation for the distribution of the maria.

Kuiper postulates that the planetesimal which produced the Hyginus crater initiated a crack some hundreds of kilometers long in a rigid crust of the moon, a crust which was floating in some way on melted silicates. Subsequently, according to Kuiper, these liquid silicates rose in the crack to form a level floor. Kuiper's solid lunar crust must have had most unusual properties, lower densities than the liquids and great tensile strength, in order for this floating block to exist under great tension and form a crack some 5–10 km. wide when "triggered" by the fall of a planetesimal. Silicates have low tensile strength even in mechanically perfect samples, and the surface of the moon subjected to enormous forces by an intense bombardment would have the strength of sand. Gilbert[5] accounted for this floor of the Hyginus Cleft by postulating partial filling of the crack with liquid silicates produced during the collisions that formed the lunar maria. It seems to me that Gilbert's explanation is at least in accord with physical possibilities, while Kuiper's is not.

There are other doubtful points in Kuiper's paper. I recently discussed the question of the origin of tektites.[9] Kuiper suggests that they originated from the moon, a very old suggestion, made again recently by Nininger.[10] It is difficult to

understand their chemical composition and terrestrial distribution unless they are of terrestrial origin.

Melted lunar silicates might be more dense than solids with which they are in equilibrium, or less dense, but they should not sometimes rise in cracks only to a certain distance below the surface and at other times rise to the tops of high mountains, as Kuiper assumes. Also it seems most improbable that nearly all the lunar surface was subjected to intense bombardment and yet substantial areas were missed completely, so that an earlier accreted surface is preserved through the time of assumed melting and bombardment, as Kuiper assumes for the region north of Mare Imbrium. Gilbert[5] maintained that vestiges of old craters could be seen in these areas and that they were covered with debris during the Imbrian collision, which seems to be a more probable explanation.

Kuiper postulates that the Alpine Valley is a crack produced during the Imbrian collision and not a groove produced by an iron-nickel object plowing through the lunar surface, as the writer has suggested. It may be that this is the case. As he points out, it has a slight bend in it, which is evident in good photographs; but it is remarkably straight for a fracture, and, if it was produced in this way, one would expect other fractures in its neighborhood. If it was produced by a missile, one wonders where the displaced material is. One also wonders whether such a large crack, some 130 km. long and 10 km. wide, could be made in material of low tensile strength. Isn't the problem similar to Hubbert's[11] problem of lifting the state of Texas? The lava in either case could have flowed into the valley from Mare Imbrium and need not have risen from the interior, since the large mountainous masses probably did not completely close the entrance to the mare. The Alpine Valley is 10 km. wide and, if it is a crack, must have been equally deep when it was formed. In this case slumping should have occurred, causing irregularities in the two sides, and any corresponding features on the two sides as noted by Kuiper must be accidental.

Kuiper believes that the central peaks of the craters cannot be due to a rebound of the lunar material after a collision. He states: "While one can visualize a rebound in a liquid or plastic, there seems no reason to suppose that a solid can act in this manner." I find that many physical scientists, particularly those conversant with ballistic problems, have observed that solids flow under high pressure. Gilbert in 1893 specifically pointed this out. On the other hand, Kuiper speaks of "squeezed-up" lavas, an idea which this writer is wholly unable to entertain as a serious physical postulate. Lavas are liquid, not putty, and could hardly form great, simple "functional" structures. And what sort of mechanism squeezed them up in the center of the craters? Kuiper's suggestions are similar to Shaler's,[12] which I studied and rejected five years ago.

No clear picture of the order of events or the times at which they occurred emerges from Kuiper's paper. Probably such a picture is impossible unless some definite conclusion has been reached in regard to the amounts of the radioactive elements in the substance of the moon, and Kuiper does not state what he believes these amounts to be. In fact without specifying such data, one cannot estimate the time required for melting or, indeed, whether melting would occur at all. Kuiper does assume that objects suitable for producing the meteorites melted 4.5×10^9 years ago, owing to radioactive heating. Hence substantial objects, some hundreds

of kilometers in radius perhaps, must have accumulated some 5 or 5.5×10^9 years ago, acquired a high temperature during $0.5–1.0 \times 10^9$ years, as estimated by Urey,[7] been broken up by some collision processes, and the debris accumulated in the chondritic meteorites, which have remained essentially unchanged for the last 4.5×10^9 years. But when did the earth and moon accumulate? Kuiper believes that the moon did not melt until some 5×10^8 years after the earth and moon were formed.

It would be a thankless task to review adequately this paper in all details. Features of the moon that have been observed and recorded during the last century are again described as though they were newly observed and then are followed by a discussion of the completely melted moon theory. On page 1104 there is a long paragraph listing facts well known for many years, and then at the bottom of the page is the statement: "It is at first somewhat surprising that the mountain ranges around Mare Imbrium should have formed during the period of maximum melting, at which time the crust must have been fairly thin and possibly was not able to support the extra load." It is certainly surprising to this reader. Then comes a discussion of the Alpine Valley that still leaves me at a complete loss as to how these massive mountains were supported.

It seems to me that Kuiper's theory is not internally consistent. Perhaps a theory of the moon which assumes that it was completely molten in its early history can be devised, even though he has not done so. Such a theory would probably best assume that the irregular shape of the moon is due to a frozen tidal wave and that the thermal conductivity of lunar material is considerably higher than current estimates for terrestrial materials. Then it should be assumed that the irregular shape and the surface features were formed very late in its history when it had acquired a very rigid structure.

In a recent publication[7] I made an attempt to explore possible radioactive heating effects during the early history of the solar system. Very complicated and puzzling histories must be assumed. Though the process of the moon's origin was undoubtedly more intricate than anyone has the courage to imagine, it is my belief today, as it has been for some five years, that very probably the moon was accumulated at low temperatures from a primitive dust cloud of solar composition with the iron in oxidized states and that the concentrations of radioactive substances within the moon are sufficiently low that melting has never occurred. In fact, I believe that present temperatures are so low that the interior of the moon has a high strength and that such low temperatures require the moon to have been formed at low temperatures and never to have been melted at any time.

* Presented at a conference on the abundances of the elements held at Williams Bay, Wisconsin, September, 1953.

[1] G. P. Kuiper, these Proceedings, **40**, 1096, 1954.
[2] H. C. Urey, *Geochim. et Cosmochim. Acta*, **1**, 209, 1951.
[3] H. C. Urey, *The Planets* (New Haven: Yale University Press, 1952).
[4] R. S. Dietz, *J. Geol.*, **54**, 359, 1946.
[5] G. K. Gilbert, *Bull. Phil. Soc. Washington*, **12**, 241, 1893.
[6] C. Patterson, *Geochim. et Cosmochim. Acta*, in press.
[7] H. C. Urey, these PROCEEDINGS, **41**, 127, 1955.
[8] G. P. Kuiper. these PROCEEDINGS, **39**, 1153, 1159, 1953.
[9] H. C. Urey, these PROCEEDINGS, **41**, 27, 1955.

[10] H. H. Nininger, *Sky and Telescope*, **2**, No. 4, 12–15; No. 5, 8–9, 1943.

[11] M. K. Hubbert, *Bull. Am. Assoc. Petroleum Geol.*, **29**, 1530, 1945.

[12] N. S. Shaler, *A Comparison of the Features of the Earth and the Moon* "Smithsonian Contributions to Knowledge," Vol. **34**, No. 1438 [1903].

EFFECT OF ADENINE MONONUCLEOTIDES ON NEURAL TUBE FORMATION OF FROG EMBRYO

By Elisabeth Ambellan

ZOÖLOGY DEPARTMENT, COLUMBIA UNIVERSITY, NEW YORK, NEW YORK

Communicated by L. C. Dunn, May 10, 1955

This is the first of a series of studies now in progress on the biochemical nature and energy relationships of developmental processes of *Rana pipiens* treated with solutions of materials thought to be essential growth substances. This treatment has the advantage of identifying effective agents without artifacts of graft or transplant experiments that may confuse inducing materials with products of cytolysis, and avoids the necessity of extrapolating from what "breis" or explants may do under experimental conditions to what the intact embryo does under relatively normal conditions.

Frog embryos require little energy during cleavage, blastulation, and early gastrulation,[1] after which energy demands are markedly increased. This period is marked biochemically by the synthesis of purines,[2] ribonucleic acids,[3] and new specific proteins,[4] and these are presumably connected with ensuing morphological processes. The measurable respiratory rate of the gastrula increases, and there is a high local concentration of adenosine triphosphate (ATP) in areas of synthetic activity.[5]

One method for synthesis of the nucleic acids and new proteins that precedes visible differentiation requires, besides the necessary raw materials and enzymes, high energy sources to build larger molecules. The first experiments, therefore, were to test whether or not the growing embryo could utilize ATP as an exogenous source of energy in furtherance of normal synthetic activities.

Methods.—Ten to twenty dejellied eggs per Stender dish in about 15 ml. of freshly made solutions were kept in an 18° C. constant-temperature bath. Solution volumes and number of eggs were constant within any one experiment. Dishes were taken out of the water bath for observation one at a time (in later experiments) and were never out of the bath for more than a few minutes. Selection of an apparently homogeneous starting population by eliminating any abnormal eggs and any that were ahead of or behind the average stage of development for the clutch resulted in 90–100 per cent of eggs within dishes maintaining the same rate of development throughout most of the experiments.

Estimations of growth rate were first made by arbitrary substaging of Shumway stages (as pictured in Rugh's manual), but in the last three experiments direct time measurements were made. Samples of experimentals were fixed in Bouin's solution, and the time taken for controls grown in 10 per cent Ringer's solution to reach the same degree of neural tube closure as the fixed experimentals was considered the measure of advanced growth rate of experimentals over controls.

Schwartz (chromatographically pure) disodium salt of adenosine triphosphate ($Na_2 \cdot ATP \cdot 3H_2O$) and Schwartz yeast adenylic acid (AMP-3) which have a pH of about 3.6 in 10 per cent Ringer's solution were brought to pH 5.6 with "Ringer's hydroxides," a mixture of Na, K, and Ca hydroxides with the same cation concentrations as Ringer's solution. Sigma adenosine diphosphate (ADP) having neutral pH in 10 per cent Ringer's solution was brought to pH 5.6 with 0.1 *N* HCl. Inorganic pyrophosphate ($Na_4P_2O_7 \cdot 10H_2O$), H_3PO_4, Schwartz adenine and Schwartz adenosine were used in various experiments.

Results.—Eighteen separate experiments involving treatment of *R. pipiens* embryos with ATP in 10 per cent Ringer's solution at pH 5.6 for short time periods beginning during late blastula stages resulted in precocious formation of neural tubes that developed 2–3 days after removal of embryos from treatment solutions. Duration of treatments ranged from 12 to 24 hours, and, when the treatments were applied at stages 7 to late 9, the effects were consistent and specific for this one morphological process. When controls were at late neural fold stage (Shumway, late 14) with open anterior folds 0.6 mm. wide, the neural ridges in experimentals were completely fused and rotation was observed (Shumway, late 15). The average advance in growth rate at this period of development was estimated by substaging methods at about 4 hours for all eighteen experiments; in the last three experiments in which this time was actually recorded, one showed $3^3/_4$, another 4, and another $4^1/_2$ hour's advance. This advanced development continued through the next neural tube stage (Shumway, 16), gradually diminished, and by tail bud stage (Shumway, 17) the treated embryos usually exhibited typical abnormalities of open blastopores, microcephaly, bloated bellies, and stunted bodies.

Growth effects, toxicity, and abnormalities were generally found in proportion to concentration of materials, length of treatment and developmental stage at which given. Treatment of blastulae at pH 5.6 with 0.008 *M* ATP was toxic in three experiments and was not used again; 0.004 *M* was toxic to some clutches, or there were many casualties within experimental dishes; 0.001 *M* sometimes showed less advance, and 0.0005 *M* showed little or no advance over controls. In general, 0.002 *M* gave the maximum advance in neural tube formation without early toxicity or later abnormalities and was used as the additional control in experiments testing effects of other materials where negative results were anticipated.

Other experiments, not included in the eighteen covered by this report, showed that continuous ATP treatment in similar concentrations never produced additional growth increase beyond that obtained by short-term treatment for the same clutch. The continuous treatment was more toxic, and the few survivors showed typical but exaggerated abnormalities in proportion to concentration. In another experiment, 0.0001 *M* ATP was effective with 8 hours of treatment applied right after first cleavage, although 0.002 and 0.004 *M* were toxic when applied this early. One experiment with *R. sylvatica* embryos showed that ATP, 0.002 *M*, pH 5.6, had the same effect in neural tube formation as with the *R. pipiens* embryos.

The exact limit of effective pH ranges of ATP was not determined, but those used in most of the experiments ranged from pH 5.5 to 5.8. At pH 3.6 both controls and experimentals died. In one experiment ATP was effective at 0.001, 0.002, and 0.004 *M* at pH 5.6, but at pH 7.1 the two higher concentrations were toxic, and the 0.001 *M* had only slight positive effects, followed by extreme abnormalities.

Experiments showed that additional Na ion concentration (equivalent to the Na_2·ATP added), inorganic phosphate, inorganic pyrophosphate, adenine, and adenosine in concentrations 0.001–0.008 *M* at pH 5.6 had no effect on neurulation.

Three separate experiments using adenosine diphosphate (ADP) at pH 5.6 resulted in the same rate of neural tube formation as with ATP treatment, and the embryos could tolerate twice the concentrations without cytolysis or abnormalities. Three experiments using yeast adenylic acid (AMP-3) in 0.001–0.004 *M* concentrations resulted in the same rate of increase in neurulation, but after effects at tail bud stage varied between clutches. Results of one experiment with a single clutch of embryos treated for 20 hours beginning during late blastula and then returned to 10 per cent Ringer's solution (minus bicarbonate and phosphate) are given in Table 1. Stages in the second column were simultaneous observations. "Hours advance" was time recorded at 18° C. for controls to reach the same stage of neural tube closure as a typical experimental fixed in Bouin's solution as a sample.

TABLE 1

EFFECT OF SHORT-TERM TREATMENT WITH ADENINE MONONUCLEOTIDE SOLUTIONS ON FROG DEVELOPMENT

Material in Solution	Shumway Stage at One Observation	Hours Advance over Controls at Neurulation (Approx.)	Observations at Tail Bud Stage	Mm. Length after Hatching
10 per cent Ringer's (controls)*	Late 14	...	Normal	5.4 av.
ATP 0.004 *M*	Late 15	4 1/2	Stubby bodies, microcephaly†	4.9
ATP 0.002 *M*	Late 15	4 1/4	Stubby	4.7–4.9
AMP-3 0.004 *M*	Late 15	4 1/4	Stubby‡	4.9
AMP-3 0.002 *M*	Late 15	4 1/4	Stubby‡	4.9
ADP 0.008 *M*	Late 15	4 1/2	Normal	5.4
ADP 0.004 *M*	Late 15	4 1/4	Normal	5.4
ADP 0.002 *M*	Late 15	4 1/4	Normal	5.4–5.6
ATP 0.004 *M* + AMP-3 0.004 *M*	Late 15	4 1/2	Normal	5.4
ATP 0.002 *M* + AMP-3 0.002 *M*	Late 15	4 1/4	Normal	5.4
ATP 0.001 *M* + AMP-3 0.001 *M*	Late 15	4 1/4	Normal	5.4

* Three replicates of controls dishes had no apparent variability in development between dishes or in individual eggs within dishes.

† Only 3 eggs were left in this dish at this stage; others died during previous treatment. Ten eggs in each of all other control and experimental dishes lived until hatching, except those fixed in Bouin's solution during experiment.

‡ In a later experiment these tail bud stage abnormalities following advanced neurulation did not appear.

Discussion.—The effects of the adenine mononucleotides, ATP, ADP, and AMP-3, were found to be generally proportional to concentration, were effective long after embryos were removed from treatment solutions, and were highly specific for one morphological process. It is assumed, therefore, that there was penetration of these materials that affected a specific metabolism of the embryos. While the effectiveness of materials applied was measured in terms of closure of the neural tube, such organized morphogenetic movements must result, in turn, from previous synthetic cellular activities and must be an integral part of the constructive pattern of development that constitutes "growth" during this period. Since these mononucleotides enhance a normal morphogenetic process, it is suggested that they can be utilized in the normal biochemical processes causing this directed morphology. The advanced neural tube formation resulting from treatment with all three mononucleotides indicates that it is the adenylic acid common to all of them that is the effective agent—adenine, adenosine, inorganic phosphate, and pyrophosphate having been experimentally eliminated from consideration when they

had no effect on development under similar conditions. It may be that the effective materials found here are used as raw materials for synthesis of nucleic acids and new proteins in a manner similar to the role of the mononucleotides suggested by Brachet[6] in neural plate formation. Essential enzymes for this synthesis, or the ability to induce them when presented with the proper substrate, are assumed to be present in the embryo, since they are needed for development under normal conditions. If other mononucleotides or combinations of them are found to have the same effect, this will lend support to Brachet's hypothesis.* In any case, this hypothesis can now be investigated further, since the specific metabolic pathway taken from the time the adenine mononucleotides are first introduced to their final effect in neural tube formation can be followed by isotope, bio-assay, or cytochemical methods.

If the adenylic acid is the effective part of these molecules, and not the high-energy pyrophosphate bond in ATP, as was first thought, these experiments indicate that both energy mechanisms and nucleic acids play a role in differentiation. In the experiment reported in Table 1, 0.004 *M* ATP was highly toxic alone but was not toxic when mixed with equimolar AMP-3. It is possible that the equilibrium found in experiments, in vitro,

$$2\text{ADP} \rightleftarrows \text{ATP} + \text{AMP}$$

is here serving to maintain the energy equilibrium required for normal growth in the entire organism.

The observed pH dependence of the ATP reactions could be explained as being due to higher penetration of less ionized molecules (expected with ATP at the more acid pH), but this is not likely here, since toxicity of ATP at pH 7.1 indicates penetrance at this pH also. Experiments with other materials used in similar or higher concentrations that had no effect on development showed that additional salts in the medium were not by themselves toxic. It may be that there is active transport by, or activation of, enzymes that function at this lower pH, but there is no evidence on this point. This pH dependence may explain Brachet's results[7] finding that ATP, AMP, and other nucleic acid derivatives were toxic for gastrula cells, "although the pH of the solution was carefully adjusted." It is possible that his solutions were adjusted to a more alkaline pH. It is also possible that there is a difference in behavior between the intact organisms used here and the explants used by Brachet.

Summary.—In twenty-four experiments *R. pipiens* blastulae treated with solutions of ATP, ADP, or AMP-3 (with specific conditions of pH and concentrations) for short periods of time resulted in precocious closure of neural tubes about 4 hours (18° C.) ahead of controls. In one experiment with *R. sylvatica*, using ATP, the results were identical. The effects of solution treatments appeared 2–3 days after embryos were returned to normal medium. Adenine, adenosine, inorganic phosphate, and pyrophosphate in similar experimental conditions had no effect on neurulation, nor had the effect of pH alone. A tentative hypothesis for the method of utilization of mononucleotides is presented.

* A recent experiment using separately cytidylic, uridylic, and guanylic acids at pH 5.6, 0.004M for 35 hours of treatment indicates that these other mononucleotides are also effective in accelerating neural tube formation.

Solution treatments that *enhance* a normal growth process offer a simple system for study of biochemical factors of morphogenesis in an intact organism under comparatively normal conditions. The mechanism by which the adenine mononucleotides, or other materials that may be found to favor a specific growth process, affect metabolism after penetration can be followed further by more refined techniques.

I wish to express thanks to Professor Lester G. Barth and Dr. Lucena J. Barth for numerous suggestions, helpful criticisms, manuscript revisions—in short, for invaluable hours of time given to this work.

[1] L. G. and L. J. Barth, *Energetics of Development* (New York: Columbia University Press, 1954).

[2] S. Graff and L. Barth, *Cold Spring Harbor Symposia Quant. Biol.*, **6**, 103–108, 1938.

[3] M. Steinert, *Biochim. et Biophys. Acta*, **10**, 427–431, 1953, quoted by Brachet, *Arch. Biol.*, **15**, 44, 1954.

[4] Barth and Barth, *op. cit.*, p. 70.

[5] J. Brachet and H. Chantrenne, *Acta Biol. Belg.*, **4**, 455, 1942, quoted by Brachet, *Chemical Embryology* (New York: Interscience Publishers, 1950), p. 403.

[6] J. Brachet, *Cold Spring Harbor Symposia Quant. Biol.*, **13**, 18–25, 1947.

[7] *Ibid.*

---

*A RAPID METHOD FOR VIABLE CELL TITRATION AND CLONE PRODUCTION WITH HeLa CELLS IN TISSUE CULTURE: THE USE OF X-IRRADIATED CELLS TO SUPPLY CONDITIONING FACTORS**

BY THEODORE T. PUCK AND PHILIP I. MARCUS

DEPARTMENT OF BIOPHYSICS, FLORENCE R. SABIN LABORATORIES, UNIVERSITY OF COLORADO MEDICAL CENTER, DENVER

Communicated by O. H. Robertson, May 5, 1955

Studies of many aspects of the genetics and metabolism of animal cells are seriously handicapped by lack of a simple, effective technique for large-scale colony production from single cells. Thus it is not easily possible to develop a new population from the single-cell survivors isolated after the action of high-energy irradiation, virus invasion, or any of a variety of toxic drugs or nutritional stresses. This difficulty also makes almost impossible the preparation of mutant cell strains, a technique which has so enormously advanced studies of cell and virus genetics and biochemical transformations in microbial systems.[1] In addition, it is not possible to determine what fraction of any animal cell population is able to initiate multiplication, and therefore it becomes difficult to evaluate the mechanism of action of agents which change growth rates.

Earle and his associates[2] have studied quantitatively the minimum number of L-strain fibroblasts required in a given volume of medium in order to insure growth. Their careful investigations have emphasized the necessity for "conditioning" by the living cells of the nutrient media currently employed in tissue culture before multiplication can be supported. They were able to produce clones from a variety

of cells by a fairly exacting technique involving sealing of individual cells in capillary tubes, where diffusible metabolic products essential for multiplication remain in association with the cells. However, at most, only a few per cent of the single cells isolated by this method could be carried to the point of self-sustaining colony development. In attempting to develop a rapid method for growing clones from every viable cell, as is routinely accomplished with bacteria, we were impressed by the demonstration of Earle and his colleagues of the need to conserve diffusible cellular products, and so directed our efforts toward furnishing isolated cells with maximal amounts of these substances.

METHODS AND MATERIALS

The HeLa strain of epithelial cell obtained by Gey *et al.*[3] from a human cervical carcinoma was used throughout. These cells, initially supplied by the George Washington Carver Foundation, are maintained in this laboratory in continuous subculture. Growth medium of the following composition was employed: synthetic solution of amino acids, vitamins, and other growth factors,[4] 40 per cent; horse serum, 10 per cent; human serum, 20 per cent; Hank's solution, 30 per cent. Penicillin (100 units/cc) and streptomycin (100 μgm/cc) were routinely added. Cultures were incubated at 36.5° C. in an atmosphere of water-saturated air containing 3–5 per cent CO_2.

Suspensions of HeLa cells were prepared by trypsinization, according to standard techniques,[5] of the stock cultures which are grown on glass. Cell counts were made with a hemacytometer, and suspensions were diluted in nutrient medium to provide the requisite cell densities. These standardized suspensions were used in two types of experiments. In one procedure a series of approximately globular droplets, with volumes varying from 0.02 to 0.001 cc. and with progressively varying numbers of cells, was deposited on glass microscope slides, which were then placed in a Petri dish, empty except for cotton pledgets saturated with the growth medium, for humidity maintenance. These dishes were incubated, and the droplets subjected to daily microscopic count of cell numbers, from which growth rates were determined. In the other procedure, 0.05 cc. of standard cell suspensions was pipetted onto a piece of microscope slide (1.0 × 2.5 cm.) and spread with the tip of the pipette over the entire surface, to insure separation of the single cells. Microscopic examination of such slides confirmed that the cells so deposited lodge at widely separated points and in numbers to be expected within the uncertainty of sampling error. The slides were incubated for 5 hours, during which the cells attached to the glass. They were then placed in Petri dishes containing nutrient medium and nonmultiplying feeder cells, as described below, and reincubated. Fresh nutrient was exchanged for the old medium after approximately 3–4 days.

EXPERIMENTAL RESULTS

1. *Establishment of the Minimum Number of Cells Necessary To Initiate Sustained Growth in Nutrient Medium Alone.*—The first step consisted in determination of the minimum colony size which would permit uniformly successful subculture and regrowth to any desired cell number. It was found that colonies with as little as 25–50 cells could regularly be trypsinized and transferred to drops on new microscope slides or test tubes, where they would eventually yield confluent growth. By successive transfer from a slide colony to a test tube and then to a 1,000-cc. bottle, cell populations as great as 10^7 could be developed. To provide a margin of safety, the figure of 100 cells was selected as a minimum colony size from which routine subculture leading to any desired cell number could reliably be expected.

Next, experiments were undertaken to ascertain conditions under which isolated single cells would grow up to colonies of 100 or more. Varying numbers of cells, in droplets of nutrient medium of controlled size, were deposited on glass slides and incubated as described, in the thought that sufficiently small droplets might limit dilution of any essential, diffusible molecules as effectively as a capillary. However, the smallest drop size that could conveniently be manipulated contained a volume of 0.001 cc., which was still very much greater than that of the capillaries used by Earle. Such droplets regularly promoted growth past the stage of 100 cells only if they contained an initial inoculum greater than 11 cells (Table 1).

TABLE 1

GROWTH OF HeLa CELLS IN DISCRETE DROPLETS OF 0.001–0.0025 CC.

No. of Trials	No. of Cells in Original Inoculum	Per Cent of Tests Which Resulted in Cell Proliferation Past the 100 Mark*
56	1–5	18
12	6–10	66
32	11–50	100

* The criterion taken for unlimited growth was the development of a colony of 100 or more, since experience demonstrated that this number of cells could reliably be grown up to 10^7 by successive subculture in progressively larger containers.

The use of conditioned medium, as recommended by Earle, did not produce appreciable improvement. The medium was harvested after having supported a growing cell culture for 18–24 hours. Regardless of whether such conditioned medium was used full strength or diluted 1:1 with fresh medium, it produced no more success than fresh medium alone in promoting growth of small numbers of cells. It was concluded that, if the bulk of the cell population is really capable of reproducing and if the sole reason for failure of a single cell to initiate sustained multiplication is lack of a sufficient concentration of cell-manufactured, diffusible metabolites, some of these may have a half-life at 37° C. considerably less than 24 hours. Experiments were therefore designed to take account of this possibility.

2. *Growth of Clones from Single Cells Using X-irradiated "Feeder" Cells To Supply Conditioning Factors.*—Since the identity of the presumed short-lived, diffusible, conditioning factors was unknown, an attempt was made to provide a continuing supply of these by a large number of metabolizing cells which are themselves incapable of multiplication (feeder cells). If such growth-inhibited cells were arranged in close juxtaposition to a single multiplying cell, they could supply the latter with the necessary growth factors which should permit multiplication into a colony. Since high-energy irradiation operates to stop cell multiplication long before general metabolism is appreciably affected, this seemed an ideal agent for preparation of feeder cells to supply the needs of the immature clone.

Details of one experimental arrangement which gives successful results are as follows: 3×10^5 HeLa cells in 7 ml. of growth medium were pipetted into a 50-mm. Petri dish and incubated overnight to form the feeder layer. The following morning the medium was removed, replaced with 1 ml. of balanced saline, and the dishes exposed to a beam of hard X-rays (210–40 kv.; 15–20 ma.; 0.5-mm. Cu and 1.0-mm. Al filters). Doses varying between 5,000 and 40,000 roentgens appeared equally effective. After exposure, the original medium, centrifuged to remove any contained cells, was replaced and a sterile, U-shaped piece of Plexiglas laid over the feeder-cell layer. Two or three segments of glass microscope slide, on which vary-

ing aliquots of a standardized HeLa cell suspension had previously been spread, were then placed on the Plexiglas holder. The arrangement employed is shown in Figure 1.[6]

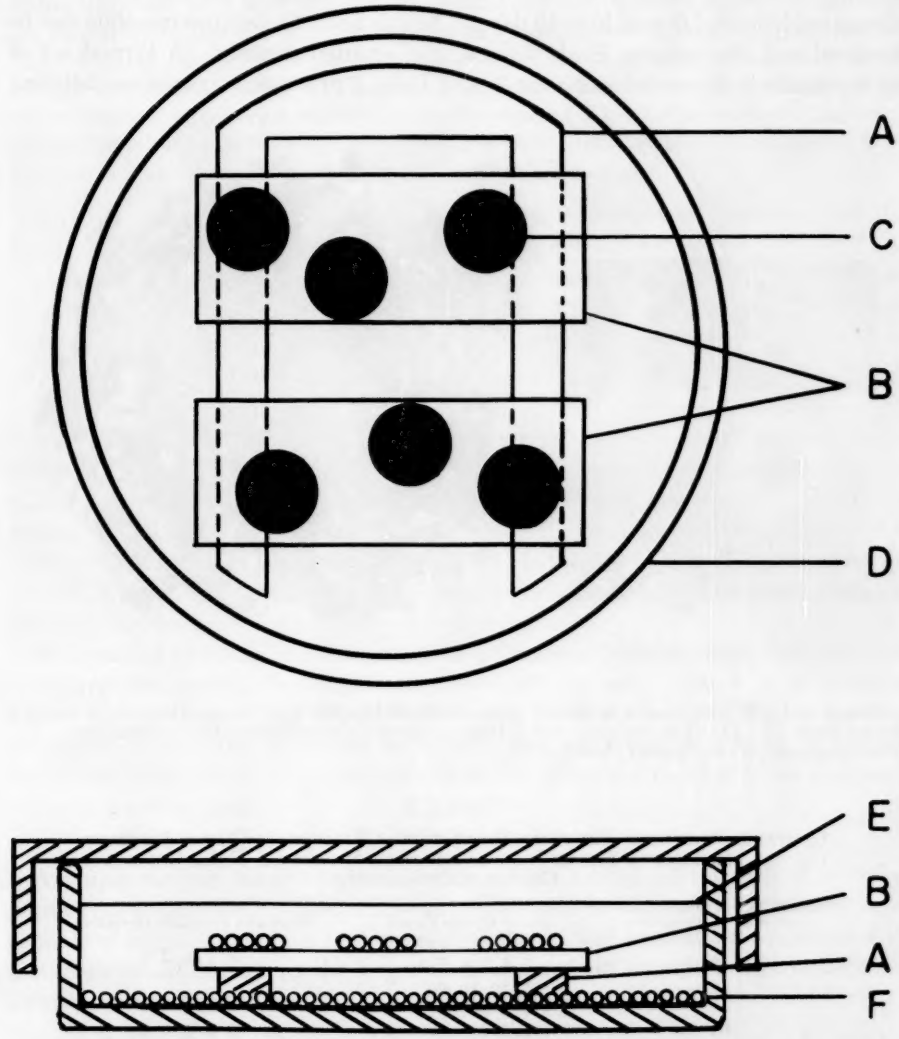
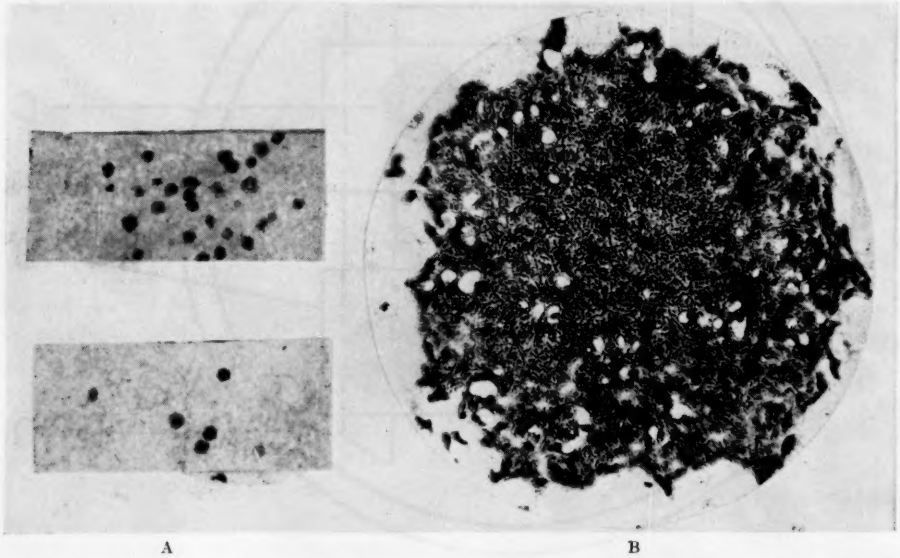


FIG. 1.—Schematic diagram of the arrangement used to grow clones of HeLa cells. Both the clones (which actually achieve a diameter of about 1 mm. in 8–10 days) and the cells are enormously exaggerated in size. *A*—U-shaped plastic holder supporting the microscope slides; *B*—segments of microscope slide on which the clones develop; *C*—clone; *D*—Petri dish; *E*—level of liquid nutrient medium; *F*—layer of feeder cells.

The plates were incubated for 8–10 days. Microscopic examination during this period revealed that the feeder layer undergoes little apparent change during the first 48–72 hours. Thereafter, this bottom cell layer degenerates, rounded forms appearing and debris accumulating in the supernate. After 4 days, only 10–20

per cent of the area of the Petri dish remains covered with recognizable cells, whereas, without irradiation, growth would be confluent.

During the static period of the feeder layer, the test cells on the microscope slide multiply steadily, reaching a clone population averaging about 800–2,000 and a diameter of about 1.0 mm. in 8–10 days. At this point the microscope slide can be removed and the colonies fixed, stained, and counted directly. A typical set of such colonies is illustrated in Figure 2, and Table 2 presents a comparison between



A B

FIG. 2.—*A*—Typical clones produced from single HeLa cells growing on glass slides above a feeder layer (×1.7). The top slide had 5 times as great an inoculum as the bottom slide. *B*—Photomicrograph of a typical clone (×50).

TABLE 2

COMPARISON BETWEEN THEORETICAL AND ACTUAL NUMBERS OF CLONES DEVELOPING ON MICROSCOPE SLIDES INOCULATED WITH SEPARATED HeLa CELLS AND GROWN AS DESCRIBED

Average No. of Single Cells Spread on Each Slide	No. of Clones Found	Mean and Standard Deviation
5	7, 3, 9, 5, 4, 2 9, 1, 2, 8, 7, 4	5.1 ± 2.8
10	10, 13, 10, 20 7, 9, 8, 10, 14 7, 5, 20 8, 9, 7, 11, 6 9, 11, 5, 6, 6, 10	9.6 ± 3.9
50	35, 44, 48 44, 43, 45, 44, 30	41.6 ± 5.6
100	77+, 68+	72.5+*

* This figure needs correction upward, because the colonies on these slides were so crowded as to make it certain that an appreciable number had been lost by coalescence.

the colony counts obtained and the average number of single cells contained in the aliquot deposited on the microscope slide. Correspondence between these numbers is as good as that obtained in bacterial assays. More than fifty such titrations

have been carried out, with no values falling outside the limits listed in the experiments of Table 2.

Experiments in which an attempt was made to secure similar results without irradiation of the feeder-cell layer were unsuccessful because cells from the bottom layer detach and are carried up to the test layer, where they may reattach and initiate colony formation. Thus, when an empty slide is placed on top of an unirradiated cell layer, as many as 150 colonies may develop which have originated from the bottom layer. Experiments which will be described later have demonstrated that the doses of X-irradiation employed inactivate the multiplication of 99.9 per cent of the feeder cells, so that, even if an occasional cell from the bottom finds its way to the test slide, it does not form a colony.

Periodic counts of the cells in the developing clones permit estimation of the doubling time, which was found to be 20 hours in young clones.

Preliminary results essentially similar were obtained with feeder cells produced by ultraviolet irradiation. However, unless quartz covers are available, the necessity for uncovering the Petri dishes during irradiation increases the risk of contamination.

DISCUSSION

The success of this titration permits the conclusion that almost all the HeLa cells obtained by trypsinization of a growing monolayer are capable of reproduction in an appropriate medium. The effectiveness of the feeder-cell principle also lends support to the idea that the sole reason for failure to obtain growth from single HeLa cells in the presence of large quantities of liquid is the loss of a short-lived, diffusible factor. New designs of simple growth arrangements for single cells, some of which eliminate the need for feeder cells, are being tested.

Since clones of these cells can easily be produced which exceed the critical size necessary for transfer to tubes and eventually to large bottles, it is possible to produce the large clonal populations needed for many types of genetic studies. Experiments with this technique on genetic and physiological actions of several agents are in progress. The applicability of this procedure to other types of animal cells is also under study.

SUMMARY

A simple method for production of clones from almost all the cells of a HeLa population is described. The technique can be used to titrate the number of cells capable of reproduction in a suspension and should find application in genetic and metabolic studies involving the action on animal cells of viruses, drugs, nutritional stresses, and mutagenic agents.

* Contribution No. 31. This study has been aided by a grant from the National Foundation for Infantile Paralysis and from the Damon Runyon Fund.

[1] J. Lederberg, *Microbial Genetics* (Madison: University of Wisconsin Press, 1951).

[2] W. R. Earle, J. C. Bryant, and E. L. Schilling, *Ann. N.Y. Acad. Sci.*, **58**, 1000, 1954.

[3] G. O. Gey, W. D. Coffman, and M. T. Kubicek, *Cancer Research*, **12**, 264, 1952.

[4] The composition of this solution, containing fewer constituents than Medium 199 (J. F. Morgan, H. J. Mortan, and R. C. Parker, *Proc. Soc. Exptl. Biol. Med.*, **73**, 1, 1950), was kindly supplied by Dr. Charity Weymouth, through Dr. Dulbecco, and will be published.

[5] Peyton Rous and F. S. Jones, *J. Exptl. Med.*, **23**, 549, 1916; R. Dulbecco and M. Vogt, *J. Exptl. Med.*, **99**, 167, 1954; J. S. Youngner, *Proc. Soc. Exptl. Biol. Med.*, **85**, 202, 1954.

[6] In our earliest experiments test cells and feeder cells were placed in the same layer. We wish to thank Dr. Leo Szilard, who suggested the more advantageous geometrical arrangement in which the test cells were placed on top of the layer of feeder cells.

*HINDERED CIS ISOMERS OF VITAMIN A AND RETINENE: THE STRUCTURE OF THE NEO-B ISOMER**

By George Wald, Paul K. Brown, and Ruth Hubbard

Biological Laboratories, Harvard University, Cambridge, Massachusetts

and William Oroshnik

Ortho Research Foundation, Raritan, New Jersey

Communicated May 19, 1955

One of the geometrical isomers of vitamin A and retinene, arbitrarily called "neo-*b*," is the precursor of the visual pigments rhodopsin and iodopsin.[1, 2] Vitamin A possesses five conjugated double bonds, to which retinene—its aldehyde—adds a conjugated carbonyl group (Fig. 1).

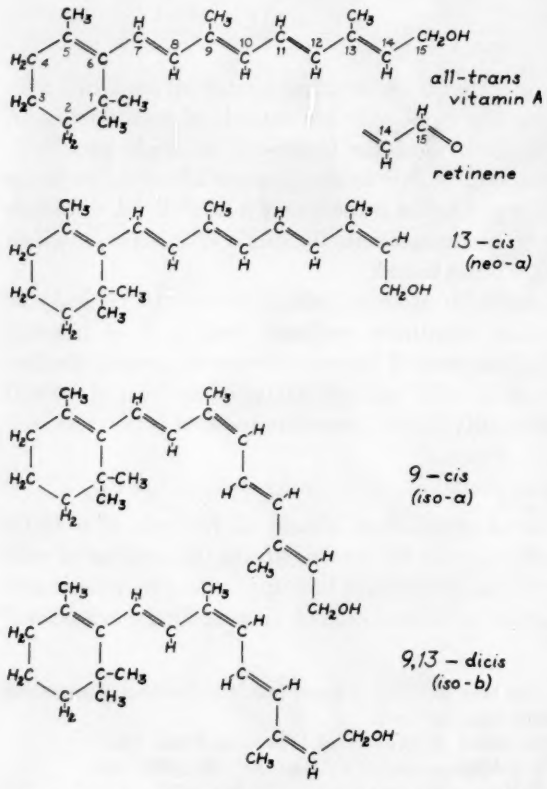


Fig. 1.—Structural formulas of the unhindered geometrical isomers of vitamin A and retinene. The *cis* linkage occurs without serious steric hindrance only at double bonds 9 and 13, and only the forms shown are relatively coplanar.

It was believed until recently that, in this class of compounds, only the double bonds from methylated carbon atoms —here numbered 9 and 13— could assume the *cis* configuration.[3, 4] At the other double bonds, the *cis* configuration encounters serious steric hindrance, necessitating a large departure from coplanarity—a large twist in the molecule. It was thought that the consequent interference with resonance should make hindered *cis* forms highly unstable. Only four unhindered geometrical isomers of vitamin A or retinene were therefore expected: all-*trans*, 9-*cis*, 13-*cis*, and 9,13-*dicis*.

Five crystalline geometrical isomers of retinene, however, have been described: all-*trans*, first prepared by Ball *et al.*;[5] neoretinenes *a* and *b*, prepared in the Harvard laboratory; and isoretinenes *a* and *b*, prepared in the Organic Research Laboratory of Distillation Products Industries in Rochester, New York (hereafter referred to as "DPI").[1, 6]

The DPI workers assigned tentative structures to three of the *cis* isomers, which they have since confirmed.[7] These are the unhindered forms, neo-*a* (13-*cis*); iso-*a* (9-*cis*), first synthesized by Graham, Van Dorp, and Arens;[8] and iso-*b* (9,13-*dicis*). Table 1 indicates the various numbering systems which have been attached to these compounds in the literature.

TABLE 1

NOMENCLATURE OF GEOMETRIC ISOMERS OF VITAMIN A AND RETINENE*

NUMBERING SYSTEM	NAME				
	Neo-*b*	Iso-*a*	Neo-*c*	Neo-*a*	Iso-*b*
Present	7-*cis*	9-*cis*	11-*cis*	13-*cis*	9,13-*dicis*
Geneva system	8-*cis*	6-*cis*	4-*cis*	2-*cis*	2,6-*dicis*
Hubbard and Wald	2-*cis*	3-*cis*	4-*cis*	5-*cis*	3,5-*dicis*

* The present paper follows Karrer's convention for numbering the carotenoids, adopted by the Organic and Biological Nomenclature Commission of the International Union of Pure and Applied Chemistry (1947), following a recommendation of the Committee on Biological Nomenclature of the National Research Council (*Chem. Eng. News*, **24**, 1236, 1946). It has the advantage of keeping the same numbers attached to corresponding carbon atoms in isoprenic polyenes of various sizes. Other workers have used the "Geneva" system.[7, 22] This proceeds from the end group and would assign different numbers to corresponding C atoms in chains of various lengths. The arbitrary numbering system formerly used by Hubbard and Wald (*J. Gen. Physiol.*, **36**, 269, 1952–1953) is also listed.

The fifth isomer, neo-*b*, must therefore possess a hindered *cis* linkage. The existence of stable polyenes containing hindered *cis* configurations has recently been placed beyond doubt by the synthesis of a number of such molecules.[9–12] Neoretinene *b* displays in its infrared spectrum evidence of a *cis* —CH=CH— linkage, necessarily hindered in this type of molecule.[13] It is the first natural compound known to possess a hindered *cis* structure.

Such a hindered *cis* linkage can occur only at double bond 7 or 11. The hindrances in these positions are shown in Figure 2. Even "unhindered" *cis* linkages

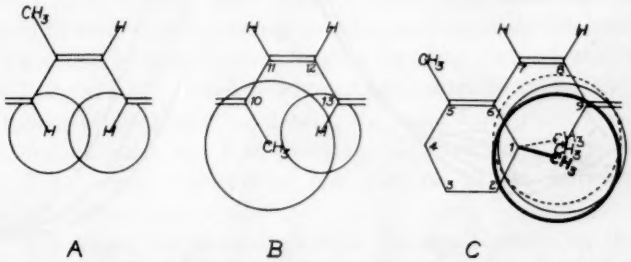


FIG. 2.—Types of steric hindrance encountered in *cis* isomers of vitamin A and retinene. Diagram *A* shows the relatively small hindrance between H atoms in such "unhindered" *cis* linkages as at double bonds 9 and 13. *B* shows the much larger hindrance between —H and —CH_3 groups involved in the 11-*cis* linkage. *C* shows the still larger hindrance between methyl groups encountered in the 7-*cis* linkage. These diagrams are drawn as though the structures were coplanar. In reality, the superpositions shown are avoided by twisting of the molecule.

involve some overlapping of —H atoms as shown in Figure 2, *A*. The small twist in the molecule that this necessitates introduces a partial block to resonance. This is believed to be responsible for the fall in extinction and displacement of spectrum toward shorter wave lengths that ordinarily accompany an "unhindered"

cis linkage. Linkages of the 11-*cis* type (Fig. 2, *B*) are much more strongly hindered, through the overlapping of an —H with a —CH_3 group. Molecules containing such linkages have been prepared synthetically.[12] The 7-*cis* linkage (Fig. 2, *C*), in which —CH_3 groups overlap with one another, involves a degree of hindrance not heretofore encountered in polyenes.

It has been suggested that neo-*b* contains an unhindered in addition to the hindered *cis* linkage—indeed, that it is the 11,13-*dicis* isomer.[13] The purpose of this paper is to show (*a*) that neo-*b* is a *monocis* isomer and hence must be either 7-*cis* or 11-*cis;* (*b*) through synthesis of the 11-*cis* isomer, that it is not 11-*cis;* and (*c*) that, for this reason and on the evidence of its own properties, neo-*b* is the 7-*cis* isomer. This structure raises an array of special physiological problems.

I. GENERAL CONSIDERATIONS

The absorption spectra of the unhindered isomers of retinene appear as solid lines in Figure 3. Zechmeister[14] has shown in plant carotenoids that a single *cis* linkage shifts the main absorption maximum in hexane 4–6 mμ toward shorter wave lengths and depresses its extinction. Two such linkages are additive in their effects.[15]

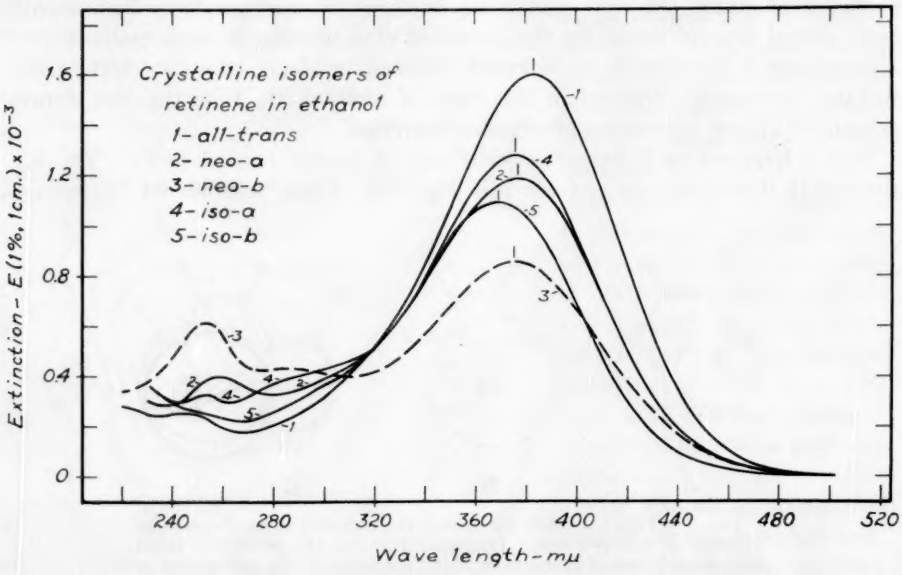


FIG. 3.—Absorption spectra of crystalline geometrical isomers of retinene in ethyl alcohol solution. The four unhindered isomers are shown with solid lines, the hindered *cis* isomer neo-retinene *b* (7-*cis*) with a broken line. The spectra exhibit three general regions of absorption: the main band, appearing at 383 mμ in all-*trans* retinene and displaced 6–7 mμ toward shorter wave lengths for each *cis* linkage; the *cis* peak region, at about 285 mμ; and subsidiary bands, at 255–260 mμ.

The unhindered retinene isomers follow this simple rule. The spectrum of the all-*trans* isomer lies at the longest wave length and possesses the highest specific extinction. The spectra of the 9-*cis* and 13-*cis* isomers lie 5–6 mμ lower in wave length (6–7 mμ in ethanol). The specific extinctions vary accordingly.[6]

The corresponding stereoisomers of vitamin A display similar relationships, with the exception of neo-*a*. In this isomer λ_{max} lies 3 mμ *higher* in wave length than in all-*trans* vitamin A and appears, therefore, to violate the ordinary rules for *cis* isomers.[15]

As Figure 1 shows, however, neo-*a* vitamin A differs from the all-*trans* configuration only in the position of the —H and —CH_2OH groups about carbon atom 14. The length and shape of the conjugated system are not thereby altered in any way. There is no particular reason for such a terminal *cis* linkage to induce the kinds of effect ordinarily associated with interior *cis* linkages. Indeed, a shift of λ_{max} to *longer* wave lengths, as in neo-*a* vitamin A, may be characteristic of terminal *cis* linkages.[12] The corresponding retinene, in which the 13-*cis* linkage is interior to the aldehyde group, displays the ordinary *cis* relationships.

As shown in Figure 1, a *cis* linkage bends the polyene chain. This creates a new electric vector at right angles to the main axis of the conjugated system. To the principal mode of electronic oscillation along the main axis, responsible for the main absorption band, there is added a smaller perpendicular component, responsible for a smaller absorption band at shorter wave lengths. Zechmeister has called the latter a "*cis* peak." Its presence is a sign of bending, its height a measure of the degree of bending. A single, centrally located *cis* linkage causes most bending and hence the highest *cis*-peak. Two *cis* linkages oppose each other in this regard: by bending the molecule in opposite directions, they straighten it somewhat, resulting in a lowered *cis* peak.[14]

In our earlier discussions of retinene spectra, we assumed that the absorption bands in the region 255–260 mμ represent *cis* peaks. Now that the configurations of these isomers are established, it is clear that the true *cis* peak region, in the sense intended by Zechmeister, lies at about 285 mμ. On this basis, as expected, the 9-*cis* isomer, being most strongly bent, has the highest absorption in this region, the 13-*cis* less, the 9,13-*dicis* isomer still less, and the all-*trans* molecule least of all.

Absorption maxima at shorter wave lengths than the *cis* peak will be referred to here as "subsidiary bands." Such bands appear commonly among the carotenoids. In terms of modes of electronic oscillation, the main band can be considered the fundamental, the *cis* peak the first overtone, and the subsidiary bands further overtones.[14] No simple correlations can yet be drawn between structural configuration and the subsidiary bands.

Hindered *cis* linkages are associated with distinct changes in the absorption spectrum. The main absorption band loses what fine structure it may possess in the all-*trans* isomer; is displaced much farther toward shorter wave lengths than with an unhindered *cis* linkage—in the type of molecule here considered, about twice as far; and is depressed much further in extinction. The *cis* peak is normal, continuing simply to indicate the bending of the conjugated system; but strong additional absorption appears in the ultraviolet, which has been ascribed to partial chromophores induced by the break in coplanarity.[12]

II. THE NEO-*b* ISOMER: MONOCIS OR DICIS?

The absorption spectrum of neoretinene *b* is shown as a broken line in Figure 3. It exhibits typical hindered *cis* characteristics in the very low specific extinction of the main band and the extraordinarily high subsidiary band. The position of the

main band, however, is anomalous; λ_{max} is the same as in the unhindered *monocis* isomers. We shall return to this point later.

It has been suggested that neo-*b* is the 11,13-*dicis* isomer.[13] This structure is incompatible with its absorption spectrum. The presence of the hindered 11-*cis* linkage alone should displace λ_{max} about 14 mμ from the all-*trans* position (see below). In neo-*b* vitamin A, a terminal 13-*cis* linkage, as explained above, might pull λ_{max} back slightly toward longer wave lengths. This factor, however, would not operate in the retinene, in which the 13-*cis* linkage, being internal to the aldehyde group, would be fully effective; and here the combination of the 11- and 13-*cis* linkages should result in a total displacement of about 20 mμ from the all-*trans* position in alcohol. Instead, one observes a total displacement of 7 mμ, as with a single, interior, unhindered *cis* linkage. On this basis alone, a *dicis* structure seems out of the question.

The isomerization experiments shown in Figure 4 lead to the same conclusion. On mild heating (3 hours in aqueous digitonin at 70° C.), neoretinene *b* isomerizes

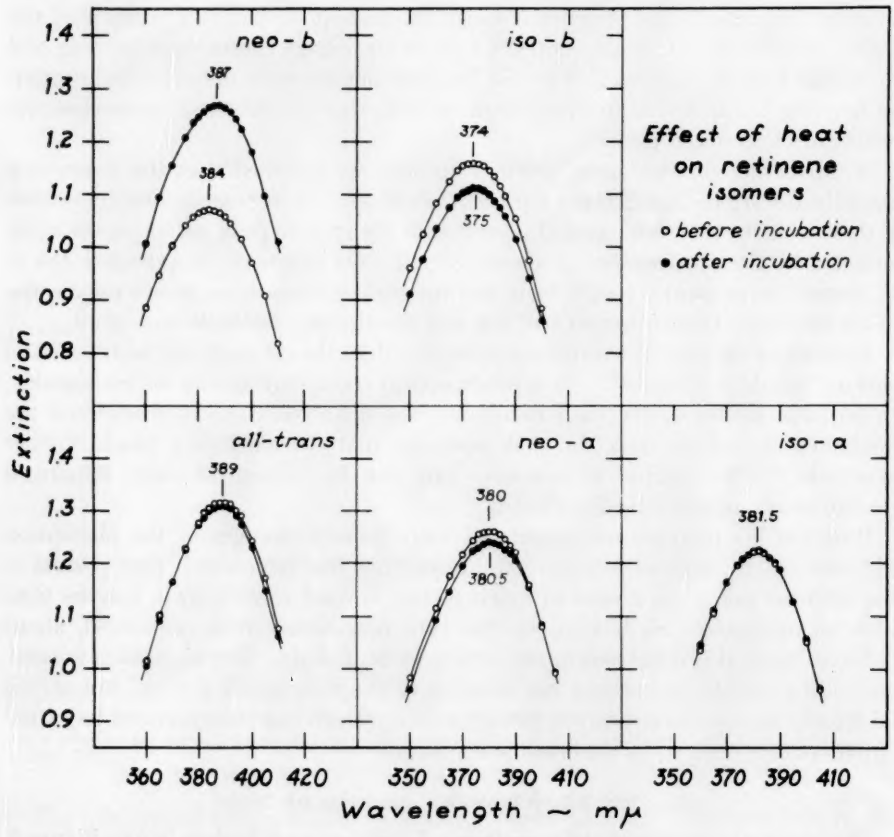


FIG. 4.—Incubation of retinene isomers in aqueous digitonin for 3 hours at 70° C. Neoretinene *b* isomerizes almost completely to all-*trans*. All-*trans* and iso-*a* retinene show no significant changes, neo-*a* and iso-*b* only slight deterioration.

almost completely to the all-*trans* configuration. All-*trans* and iso-*a* retinene remain unchanged, and neo-*a* and iso-*b* deteriorate slightly (isomerization would cause the E_{max} of these isomers to go *up*, not down as here). That is, the hindered *cis* linkage of neoretinene *b* isomerizes under these conditions, but not the 9-*cis* and 13-*cis* linkages of iso-*a*, iso-*b*, and neo-*a*. If neoretinene *b* contained any combination of a hindered with an unhindered *cis* linkage, the first product of its isomerization would be 9-*cis* or 13-*cis* retinene, which under these conditions would isomerize further only with difficulty. This is not the observed result.

The irradiation of any retinene isomer with light which it can absorb leads eventually to the formation of the same quasi-equilibrium mixture of isomers. When this reaction is carried out in such a polar solvent as alcohol or water, the final mixture contains a large fraction of neoretinene *b*.[1]

If, as suggested, neoretinene *b* contained a 13-*cis* in addition to the hindered *cis* linkage, it should be formed more readily from 13-*cis* (neo-*a*) than from all-*trans* retinene. If, on the contrary, it contains the hindered *cis* linkage alone, it should be formed more readily from the all-*trans* isomer. By taking advantage of the specific reaction of neoretinene *b* with opsin to form rhodopsin, one can measure the rate of formation of this isomer from any of the others. When neoretinene *b* is incubated in the dark with excess opsin, it forms rhodopsin with a rise of maximal extinction (E_{max}) of 1.5–1.7 times. This assay yields reliable results with as little as 10^{-4} micromole of neoretinene *b*.[6, 16]

In Figure 5 we compare by this method the rates of formation of neoretinene *b* from all-*trans* and 13-*cis* retinene. Samples of both isomers in digitonin, containing about 0.050 micromole of retinene per milliliter, were exposed to the focused light of a 160-watt microscope lamp passing through a $^1/_{10}$ neutral filter. At intervals the absorption spectra were recorded, and aliquots of 0.014 micromole of retinene were removed and assayed for neoretinene *b*. The formation of neoretinene *b* from the all-*trans* isomer starts at a high rate and follows an approximately hyperbolic course. When 13-*cis* retinene is the starting material, however, there is an initial lag, and the course of synthesis is S-shaped. At the end of the reaction both samples contain about 0.0034 micromole of neoretinene *b*, about 25 per cent of the total retinene in the sample. An experiment performed at a higher light intensity yielded the same result in shorter time.

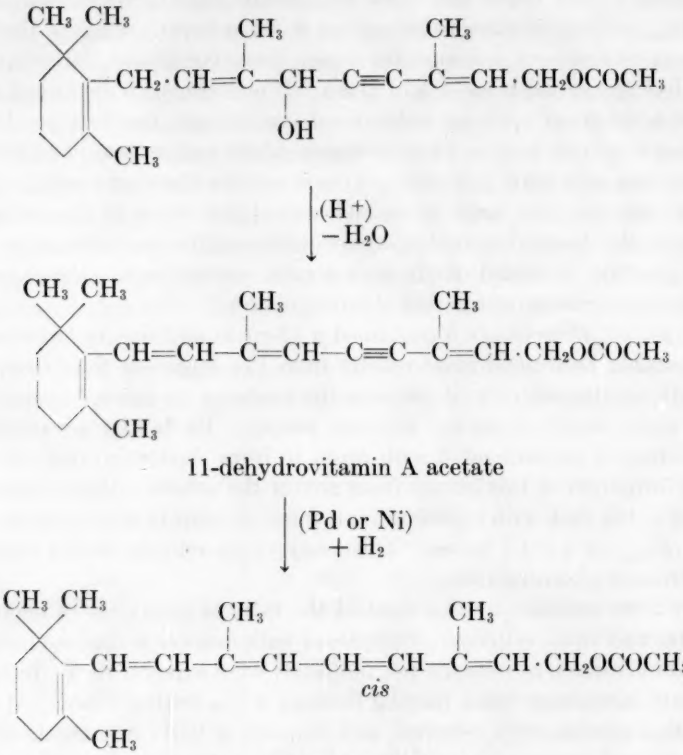
It is apparent that neoretinene *b* is formed more readily from all-*trans* than from 13-*cis* retinene. It is hardly possible, therefore, that this isomer contains a 13-*cis* linkage. On the contrary, the lag in neoretinene *b* synthesis when neo-*a* retinene is used as precursor suggests that it is necessary to isomerize the 13-*cis* linkage to *trans* before neoretinene *b* can be formed.

In summary, therefore, neoretinene *b* is almost certainly a *monocis* isomer. All-*trans* retinene appears to be its immediate precursor as well as the first product of its isomerization.

III. 11-*cis* VITAMIN A AND RETINENE (NEO-*c*)

Since the neo-*b* isomer possesses a single *cis* linkage in a hindered position, it must be either 7-*cis* or 11-*cis* (Fig. 6). The synthesis of 11-*cis* vitamin A makes it possible to choose between these alternatives. It was prepared by the following procedure, the details of which will be published elsewhere:[17]

SYNTHESIS OF 11-*cis* VITAMIN A



11-dehydrovitamin A acetate

11-*cis* vitamin A acetate

The crystalline phenylazobenzoate and β-anthraquinone carboxylate were prepared by the procedures of Isler *et al.*[18] The phenylazobenzoate has the following properties: m.p., 99–100° C.; λ_{max} in ethanol, 317.5 mμ; molar extinction (ϵ) = 51,900. The β-anthraquinone carboxylate has the following properties: m.p., 139–140° C.; λ_{max} in ether, 313.5 mμ (ϵ = 33,000) and 255.5 mμ (ϵ = 31,300).

When mixed in chloroform solution with saturated antimony chloride, 11-*cis* vitamin A yields a blue product with λ_{max} = 620 mμ, identical with that formed by other geometrical isomers of vitamin A.

The absorption spectrum of 11-*cis* vitamin A in ethanol is shown in Figure 7. It displays the typical properties associated with a hindered *cis* isomer. λ_{max} in hexane is at 311 mμ (312 mμ in ethyl alcohol), 14 mμ below that of all-*trans* vitamin A. *E* (1 per cent, 1 cm.) in ethanol is 920; ϵ = 26,200. These are by far the lowest values of λ_{max} and specific extinction found among the *monocis* isomers of vitamin A.

For comparison, the spectrum of neo-*b* vitamin A, obtained by reducing crystalline neoretinene *b*[19] with potassium borohydride, is also shown in Figure 7. It is markedly different from the spectrum of the 11-*cis* isomer.

A portion of 11-*cis* vitamin A was oxidized to the corresponding retinene by filtering through a short column of dry manganese dioxide. The retinene was chroma-

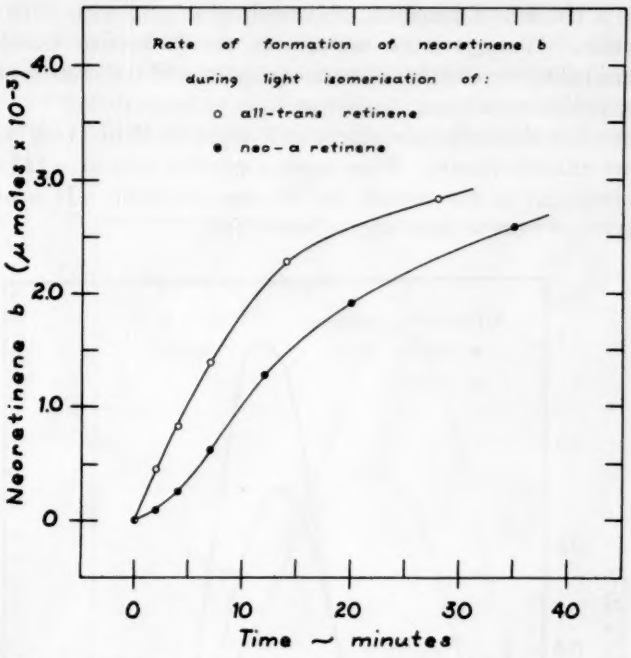


FIG. 5.—Formation of neoretinene *b* by exposing all-*trans* and neo-*a* (13-*cis*) retinene to light in digitonin solution. The rise of neoretinene *b* was measured by incubating aliquots with excess opsin, so forming rhodopsin. All-*trans* retinene isomerizes to neo-*b* at a high initial rate, as though its immediate precursor. From 13-*cis* retinene, neo-*b* forms with a lag period, presumably owing to the necessity to form all-*trans* retinene first.

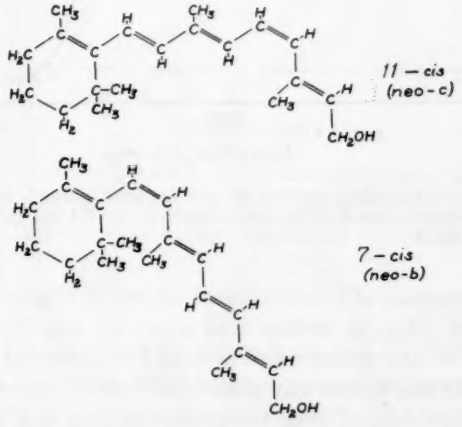


FIG. 6.—Structural formulas of the hindered *monocis* isomers of vitamin A. Both isomers are twisted as well as bent at the *cis* linkage, owing to steric hindrance, in the 11-*cis* isomer as shown in Fig. 2, *B*, and in the 7-*cis* isomer as shown in Fig. 2, *C*.

tographed on a column of alumina, weakened by mixing with 7–10 per cent its weight of water. 11-*cis* retinene appears in the lowermost fraction. It was eluted and purified by two further chromatographings of the same type. We have not yet succeeded in crystallizing this isomer.

11-*cis* retinene is sterically unstable, much more so than 11-*cis* vitamin A or than any other retinene isomer. When kept in solution even at −15° C. and dark, it isomerizes from day to day toward the all-*trans* condition. It is probable that all our preparations were to some degree isomerized.

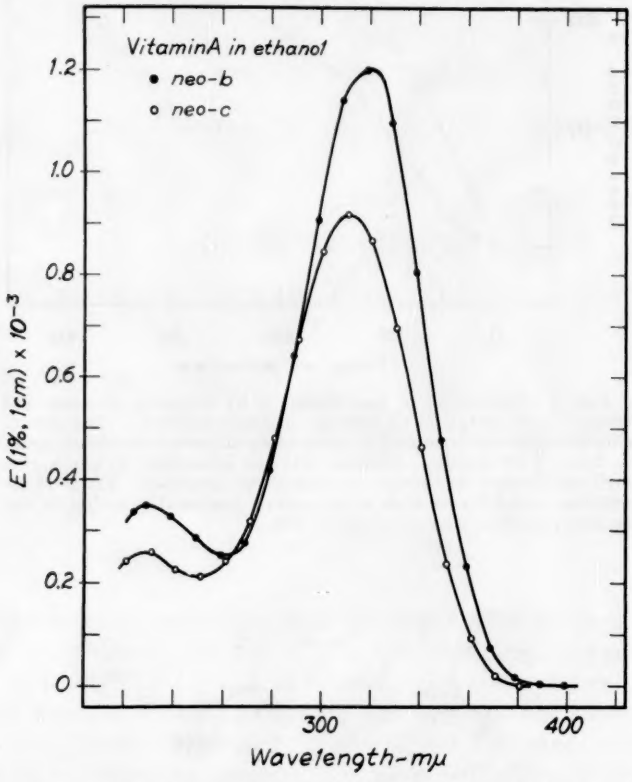


FIG. 7.—Absorption spectra of the hindered *monocis* isomers of vitamin A in ethanol. The main bands lie at 311 mμ (neo-*c*) and 321 mμ (neo-*b*).

The absorption spectrum of 11-*cis* retinene, shown in Figure 8, again is typical of a hindered *cis* isomer. λ_{max} in hexane is at about 355 mμ, 14 mμ from that of the all-*trans* isomer. The absorption band is broad and skewed by a sharply rising absorption below 310 mμ, apparently associated with a large subsidiary band. *E* (1 per cent, 1 cm.), calculated from isomerization data and comparison with the antimony chloride product, is 620–700. As in the case of 11-*cis* vitamin A, these are the lowest λ_{max} and specific extinction among the retinene isomers. The spectrum of crystalline neoretinene *b*, also shown in Figure 8, has very different properties.

A final property is of special interest. A preparation of 11-*cis* retinene was incubated in the dark for one hour with excess cattle opsin. Then hydroxylamine was added to break up fortuitous combinations of retinene with other molecules, and the product was exposed to light to bleach any photosensitive pigment that may have formed. A trace of bleaching occurred, maximal at 498 mμ, apparently

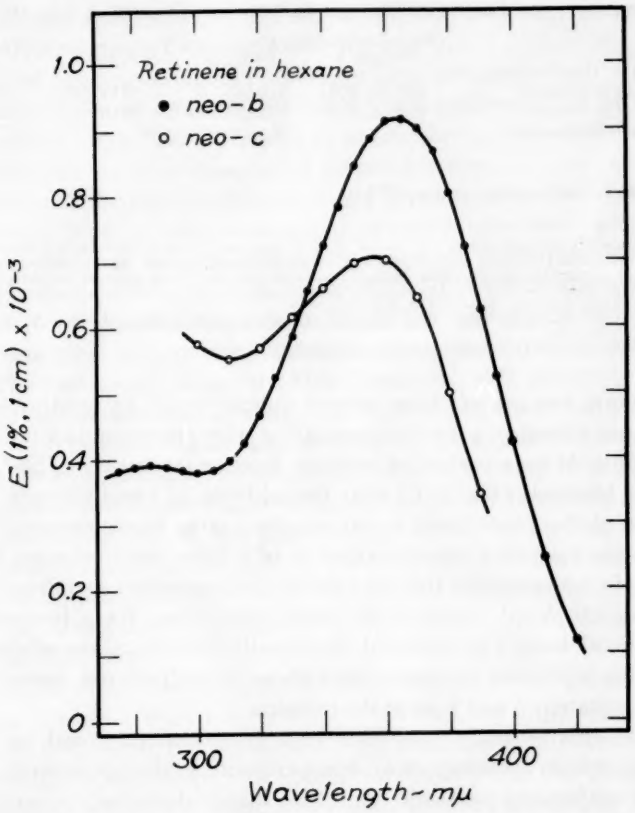


FIG. 8.—Absorption spectra of the hindered *monocis* isomers of retinene in hexane. The main bands lie at 355 mμ (neo-*c*) and 363 mμ (neo-*b*). Neo-*c* retinene is highly unstable, and this preparation may have been slightly isomerized at the time these measurements were made.

due to rhodopsin. Its E_{max} was 12 per cent of that of the retinene employed. Neoretinene *b* under the same circumstances would have yielded rhodopsin of E_{max} 150–170 per cent of the retinene employed.[16] The trace of rhodopsin obtained in this instance probably arose from a small contamination with neoretinene *b;* for in the alumina chromatogram neo-*a*, neo-*b*, and 11-*cis* retinene are adsorbed close together, in that order of descent. We conclude that 11-*cis* retinene has little or no capacity to form rhodopsin.

IV. DISCUSSION

The properties of the two hindered *monocis* isomers of vitamin A and retinene—neo-*b* and 11-*cis*—are summarized in Table 2. It is plain that these are different substances. We shall refer hereafter to the 11-*cis* isomer as "neo-*c*." As the only apparent alternative to the 11-*cis* configuration, neo-*b* must be 7-*cis*.

TABLE 2

PROPERTIES OF THE HINDERED *Monocis* ISOMERS OF VITAMIN A AND RETINENE*

	7-*cis* (NEO-*b*)		11-*cis* (NEO-*c*)	
	Vitamin A	Retinene	Vitamin A	Retinene
λ_{max} (mμ) in ethanol	320 (1,200)	376 (857)†	312 (920)	373 (700)
λ_{max} (mμ) in hexane	318 (1,170)	363 (915)	310	355
Rhodopsin formation	...	E_{max} rises 1.5–1.7 times	...	Trace?
Biopotency (per cent)	23‡	47‡	<3§	...

* The numbers in parentheses are values of E (1 per cent, 1 cm.). Rhodopsin formation was tested by incubating retinene isomers one hour in the dark with excess opsin. Biological potency is stated as percentage effectiveness (molar) for rat growth and liver storage, compared with all-*trans* vitamin A acetate (= 100).

† Dieterle and Robeson, *Science*, **120**, 219, 1954.

‡ Ames *et al.*, Abstracts, 126th Meeting of the American Chemical Society, New York, 1954, pp. 36C (*a*) and 37C (*b*).

§ Oroshnik *et al.*, *J. Am. Chem. Soc.* (in preparation).

This structure has not yet been proved unequivocally by synthesis. Attempts to make 7-*cis* vitamin A by hydrogenating the corresponding acetylene (7,8-dehydrovitamin A) by a variety of methods have so far failed. The 7,8 position is so extremely hindered (Fig. 2, *C*) that the addition of two hydrogen atoms in *cis* configuration at the triple bond would require a large rearrangement of the shape of the molecule, including the introduction of a large twist between the ring and side-chain. It is presumably this that makes hydrogenation so difficult.

Spectrum of the Neo-b Isomer.—We have noted that, for a hindered *cis* form, λ_{max} of the neo-*b* isomer is displaced abnormally little from the all-*trans* position. The spectrum is shifted no more than those of unhindered *monocis* isomers—5 mμ in neo-*b* vitamin A and 7 mμ in the retinene.

The key to this anomaly goes back to a consideration raised by Oroshnik *et al.*[9] The absorption spectrum of all-*trans* vitamin A, though generally considered typical of a conjugated pentaene, is in fact highly degraded. Compared with a straight-chain pentaene with comparable substituents, the vitamin A spectrum has lost its fine structure, is displaced some 20 mμ toward shorter wave lengths, and has an abnormally low specific extinction. The reason for this spectral degradation is steric hindrance between the 1-methyl groups and the 8-hydrogen atom, which forces the β-ionylidene ring out of coplanarity with the side-chain. Consequently, the double bond in the ring is only partially conjugated with the side-chain and makes a correspondingly small contribution to the chromophore.

For this reason a hindered *cis* linkage in the 7,8 position has much less effect than one in the 11,12 position. It only decreases further the already reduced conjugation between the side-chain and the ring double bond. The displacement of spectrum is correspondingly small, in the present instance as small as for an unhindered *cis* linkage elsewhere in the chain. It is only in the 7,8 position that a hindered *cis* linkage could have this restricted effect.

Formation of the Neo-b Isomer.—The *cis* isomers of the polyenes represent higher energy states than the all-*trans* configuration, the hindered *cis* isomers very considerably so (cf. Pauling[4]). Hindered *cis* forms are therefore hardly to be expected under conditions of true thermodynamic equilibrium. It is in this sense that one may interpret Pauling's conclusion that they should scarcely occur at all. They probably always require special conditions for their formation, including a special supply of energy. It is now clear, however, that, once formed, they are much more stable than had been anticipated.[6] Indeed, on occasion they may be more stable than unhindered *cis* isomers (cf. Oroshnik and Mebane[12]).

Two methods of forming hindered *cis* isomers have been described: irradiation with light, used initially to form neoretinene *b*;[1] and hydrogenation of the corresponding acetylene.[9] The latter process, as noted above, succeeded in the synthesis of neo-*c* vitamin A yet failed in that of the neo-*b* isomer.

On the other hand, the simple exposure of any geometrical isomer of retinene in alcohol to light that it can absorb—blue, violet, or near-ultraviolet—induces a large formation of neoretinene *b*. As Figure 5 shows, at the end of such irradiation about 25 per cent of the retinene is in this form. For so hindered a structure this seems an astonishingly high yield. It illustrates an important consideration. Among all the possible geometrical isomers of the polyenes, a few configurations tend to form selectively. The factors which govern this choice are still obscure but clearly override the distinction between hindered and unhindered forms (cf. Zechmeister and Pinckard[11]). The neo-*b* isomer appears to have such a preferred configuration.

Another condition, important physiologically, aids in forming the neo-*b* isomer. An enzyme occurs in eye tissues which catalyzes specifically the interconversion of all-*trans* and neo-*b* retinene. In the dark its equilibrium lies far over toward the all-*trans* configuration; but in dim light a quasi-equilibrium mixture is produced, containing about 35 per cent neoretinene *b*.[20] In this instance the geometry and energetics of the attachment of retinene to the enzyme protein appear to act selectively on the 7,8 linkage.

Distribution of the Neo-b Isomer in the Tissues.—The only conditions at present known to form the neo-*b* isomer involve light. These are important in the eye and may operate in the skin, but of course are not available to internal tissues.

Before its peculiar structure and the special problems which this raises were appreciated, the neo-*b* isomer was assumed to occur in liver oils and to circulate in the blood. In the first syntheses of "rhodopsin" from liver oil vitamin A, the fact was intentionally disregarded that it always possessed a lower λ_{max} than rhodopsin extracted from retinas, in the belief that at the time it was more important to learn the conditions of synthesis of this type of pigment than to hesitate over what seemed a minor disparity.[1, 21] It is apparent now that these early syntheses had yielded mainly or entirely iso-rhodopsin, so demonstrating the presence of iso-*a* vitamin A, not of the neo-*b* isomer. We are now undertaking new experiments to determine whether neo-*b* vitamin A occurs in the liver or in the blood. At present it is known to occur only in the eye.

Nutritional Significance of Neo-b Vitamin A.—Beyond its special role in vision, vitamin A fulfills some general function in the tissues, the nature of which is not understood. We know only that warm-blooded vertebrates deprived of this vitamin stop growing, deteriorate, and eventually die.

Is neo-*b* the active form of vitamin A in tissues other than the eye? A first approach to answering this question is offered by the recent bio-assays of neo-*b* vitamin A and retinene performed by Ames *et al.*[22] Compared with all-*trans* vitamin A acetate, neo-*a* is about 75 per cent active, and iso-*a*, iso-*b*, and neo-*b* about 23 per cent active, in growth and liver storage assays in the rat. On the same basis, all-*trans* and neo-*a* retinene have about 91 per cent, iso-*a* and iso-*b* about 18 per cent, and neo-*b* retinene about 47 per cent the molar activity of all-*trans* vitamin A acetate.[22] An independent bio-assay of neo-*c* vitamin A shows it to have less than 3 per cent the activity of the all-*trans* isomer.[17]

According to these data, neo-*b* has a relatively low biological potency compared with all-*trans*. Indeed, these observations imply that the all-*trans* isomer may be the form used in general by the tissues and that other isomers may be effective in promoting growth and liver storage primarily to the degree that they serve as precursors of all-*trans* vitamin A. In this sense the positions of the unhindered isomers seem clear, for the 13-*cis* linkage is known to isomerize to *trans* very readily, the 9-*cis* linkage only with difficulty. The position of the neo-*b* isomer in this regard needs further clarification, both its low potency in the vitamin A, which in vitro isomerizes to all-*trans* very easily, and the doubling of potency in the retinene. The reason for the very low activity of neo-*c* vitamin A is still more obscure. Before these bio-assays can be interpreted physiologically, two further problems need to be solved: (1) how efficiently the various isomers are absorbed from the intestine and (2) the configurations in which they are deposited in the liver and other tissues.

At present we can be certain of the occurrence of the neo-*b* isomer only in the eye tissues and of its function only as the precursor of the pigments of rod and cone vision.

V. SUMMARY

The neo-*b* isomer of vitamin A and retinene, a hindered *cis* form, is the precursor of the visual pigments rhodopsin and iodopsin. The present paper shows it to be a *monocis* isomer, therefore either 7-*cis* or 11-*cis*. The 11-*cis* isomer has been synthesized and found to possess very different properties from neo-*b*. It may be called "neo-*c*." The neo-*b* isomer must therefore be 7-*cis*. Its properties are in specific agreement with this structure. The formation of the neo-*b* isomer raises special problems chemically and physiologically. Nothing is known of its biological occurrence or function outside the eye.

* This investigation was supported in part by grants to G. W. from the Rockefeller Foundation and the Office of Naval Research. We are indebted to the Organic Research Laboratory of Distillation Products Industries for a gift to G. W. of crystalline neoretinene *b*.

[1] R. Hubbard and G. Wald, *J. Gen. Physiol.*, **36**, 269, 1952–1953.

[2] G. Wald, P. K. Brown, and P. H. Smith, *J. Gen. Physiol.*, **38**, 623, 1954–1955.

[3] L. Pauling, *Fortschr. Chem. org. Naturstoffe*, **3**, 203, 1939.

[4] L. Pauling, *Helv. Chim. Acta*, **32**, 2241, 1949.

[5] S. Ball, T. W. Goodwin, and R. A. Morton, *Biochem. J.*, **42**, 516, 1948.

[6] R. Hubbard, R. I. Gregerman, and G. Wald, *J. Gen. Physiol.*, **36**, 415, 1952–1953.

[7] C. D. Robeson, J. D. Cawley, L. Weisler, M. H. Stern, C. C. Eddinger, and A. J. Chechak, Abstracts, 126th Meeting of The American Chemical Society, New York, 1954, p. 34C.

[8] W. Graham, D. A. Van Dorp, and J. F. Arens, *Rec. trav. chim.*, **68**, 609, 1949.

[9] W. Oroshnik, G. Karmas, and A. D. Mebane, *J. Am. Chem. Soc.*, **74**, 295, 1952.

[10] C. F. Garbers, C. H. Eugster, and P. Karrer, *Helv. Chim. Acta*, **35**, 1850, 1952.
[11] K. Lunde and L. Zechmeister, *J. Am. Chem. Soc.*, **76**, 2308, 1954; L. Zechmeister and J. H. Pinckard, *J. Am. Chem. Soc.*, **76**, 4144, 1954.
[12] W. Oroshnik and A. D. Mebane, *J. Am. Chem. Soc.*, **76**, 5719, 1954.
[13] C. D. Robeson, W. P. Blum, J. M. Dieterle, J. D. Cawley, and J. G. Baxter, Abstracts, 126th Meeting of the American Chemical Society, New York, 1954, p. 34C.
[14] L. Zechmeister, *Chem. Rev.*, **34**, 267, 1944.
[15] C. D. Robeson and J. G. Baxter, *J. Am. Chem. Soc.*, **69**, 136, 1947.
[16] G. Wald and P. K. Brown, *J. Gen. Physiol.*, **37**, 189, 1953–1954.
[17] W. Oroshnik, P. K. Brown, R. Hubbard, and G. Wald, *J. Am. Chem. Soc.* (in preparation).
[18] O. Isler, W. Huber, A. Ronco, and M. Kofler, *Helv. Chim. Acta*, **30**, 1911, 1947.
[19] J. M. Dieterle and C. D. Robeson, *Science*, **120**, 219, 1954.
[20] R. Hubbard, *Federation Proc.*, **14**, 229, 1955.
[21] R. Hubbard and G. Wald, these PROCEEDINGS, **37**, 69, 1951.
[22] S. R. Ames, W. J. Swanson, and P. L. Harris, Abstracts, 126th Meeting of the American Chemical Society, New York, 1954, pp. 36C (*a*) and 37C (*b*).

ISOTOPE SEPARATION BY IONIC EXPANSION IN A MAGNETIC FIELD

BY JOSEPH SLEPIAN

WESTINGHOUSE RESEARCH LABORATORIES, EAST PITTSBURGH, PENNSYLVANIA

Communicated May 12, 1955

I. *Ionic Gas, Single Ion, in Stationary (Hypothetical) Container.*—Consider a gas at a very high temperature T, more than 100,000° absolute, or 10 electron volts or more, in a stationary (hypothetical) container, which reflects the ions and electrons which strike it completely and elastically. The gas might be a metal, with ions of mass m_1 and electrons of mass m_0. Let the ion density be n per cubic centimeter and the electron density also be n per cubic centimeter.

We will have

$$\tfrac{1}{2}m_1\overline{v_1^2} = \tfrac{1}{2}m_0\overline{u^2} = \tfrac{1}{2}kT, \tag{1}$$

where $\overline{v_1^2}$ and $\overline{u^2}$ are the mean-square velocity in any particular direction of ions and electrons, respectively, and k is Boltzmann's constant. The ratio of m_1 to m_0 is 20,000 or more. Hence the ratio of $\sqrt{\overline{u^2}}$ to $\sqrt{\overline{v_1^2}}$ is several hundred (see Fig. 1).

An ion impacting and rebounding off the wall conveys a momentum $2m_1v_1$ to the wall. An electron conveys a momentum $2m_0u$ to the wall. Thus the gas conveys a momentum $nm_1\overline{v_1^2} + nm_0\overline{u^2}$ to each second, or a force of $2nkT$ per square centimeter of wall.

Now let the walls slowly expand from volume V to $V + dV$ cubic centimeters. The work done upon the total wall is $2nkT\,dV$. This must be equal to the diminution of energy of the ions and electrons inside.

$$-d(3nkTV) = 2nkT\,dV,$$

$$-3knV\,dT = 2nkt\,dV,$$

$$\frac{dT}{T} = -\tfrac{2}{3}\frac{dV}{V} = +\tfrac{2}{3}\frac{dn}{n}, \tag{2}$$

$$\frac{T}{T_0} = \left(\frac{n}{n_0}\right)^{2/3}, \tag{3}$$

II. *Ionic Gas, Single Ion, Moving in (Hypothetical) Container, with Reflecting Side Walls.*—Let the ionic gas, with accompanying density of electrons n be streaming through the (hypothetical) container. Let the temperature of the gas entering be $T_0°$ absolute, the initial streaming velocity be v_{00} cm/sec, and the entering section be S_0 square centimeters.

Let the gas be emitted from the container with section S cm.^{2}; temperature T, absolute; and velocity v_0 cm/sec (see Fig. 2). Instead of emitting the gas from the container, we can instead let the gas precipitate on real slats. By insulating the slats, we insure that the rate of arrival of electrons equals the rate of arrival of ions or that v_0 is the same for electrons as for ions.

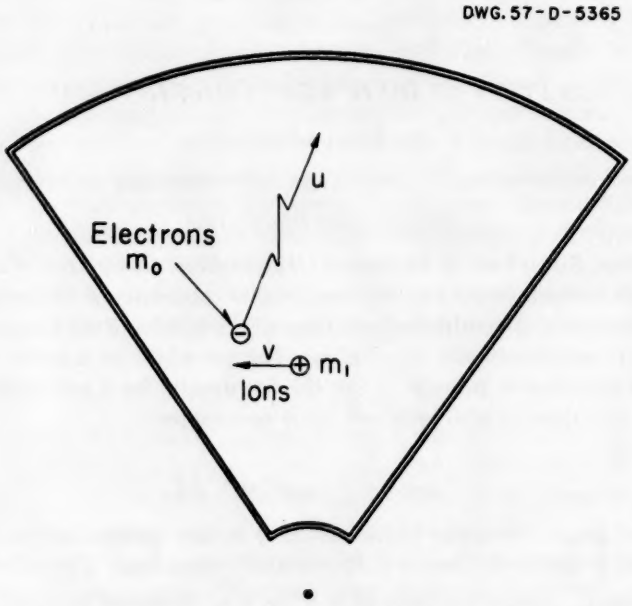


FIG. 1.—Ionic gas, stationary in (hypothetical) container with reflecting walls.

We have, for the random initial motion of ions and electrons,

$$^1/_2 m\overline{v_1}^2 = {}^1/_2 m_1\overline{u}^2 = {}^1/_2 kT_0, \text{ with } \sqrt{\overline{u}^2} \text{ several hundred times } \sqrt{\overline{v_1}^2}. \tag{4}$$

For the random final motion of the ions and electrons, we have equation (1).

There is no exchange of energy between the charged particles and the side walls, since the reflection of the ions and electrons is completely elastic. The total energy per ion is therefore constant:

$$^1/_2 m_1 v_{00}^2 + 3kT_0 = {}^1/_2 m_1 v_0^2 + 3kT, \tag{5}$$

neglecting the energy of the electron given by $^1/_2 m_0 v_{00}^2$ and $^1/_2 m_0 v_0^2$. Also, we have

$$n_0 v_{00} S_0 = n v_0 S. \tag{6}$$

Substituting equations (3) and (6) in equation (5),

$$\tfrac{1}{2}\, m_1 v_{00}^2 + 3kT_0 = \tfrac{1}{2}\, m_1 v_0^2 + 3kT_0 \left(\frac{v_{00} S_0}{v_0 S}\right)^{2/3}. \tag{7}$$

If $\tfrac{1}{2}kT_0$ is nearly equal to $\tfrac{1}{2}m_1 v_{00}^2$, then, when S_0/S is nearly equal to zero, $v_0^2 = 7v_{00}^2$, and the temperature becomes

$$7^{1/3} S^{2/3}\, T = S_0^{2/3}\, T_0, \qquad \text{or} \qquad T = 7^{-1/3}\left(\frac{S_0}{S}\right)^{2/3} T_0. \tag{8}$$

Thus the velocity of the gas approaches a limiting value $v_0 = \sqrt{7}v_{00}$, and the temperature of the gas approaches zero as $S^{-2/3}$, as S becomes infinite.

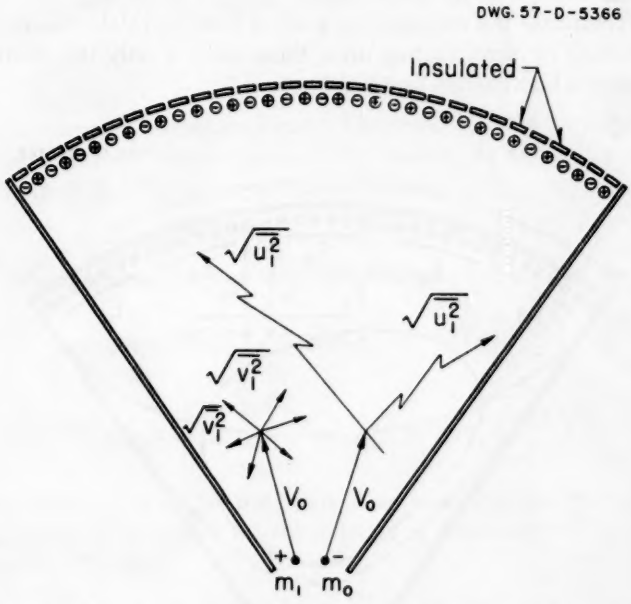


FIG. 2.—Ionic gas, moving in (hypothetical) container. Reflecting walls (hypothetical).

III. *Effect of a Magnetic Field Parallel to Side Walls, with Electric Potential Difference between Them; Other Walls of Insulating Segments; All Walls Real.*—Now let us trace the effects of magnetic field and electric field indicated in the title of this section. The motion of the ions and electrons with mean radial velocity v_0 in the magnetic field causes a mean force of $-eHv_0$ on the ions and a mean force of eHv_0 on the electrons. If the electric field produced by the potential difference on the walls is just enough to produce a mean force $+eHv_0$ on the ions and a mean force $-eHv_0$ on the electrons, then the electric field will balance the effect of the magnetic field on the uniform motion of ions and electrons. There,

$$E_\theta = -\frac{1}{r}\frac{\delta V}{\delta \theta} = Hv_0. \tag{9}$$

What will now be the effect of V at smaller radii? A current will flow in the central part from the positive plate to the negative plate. At the positive plate, a space charge of negatively charged electrons forms. These charges are in the electric field which is perpendicular to the positive plate. Under these conditions, the negative space charge moves outward parallel to the positive surface.

In the same way, at the negative plate, a positive space charge exists, which is in the electric field perpendicular to the negative plate. The net effect is that the positive ions move outward parallel to the negative surface.

It follows that everywhere where r is less than the value which makes $E_\theta = Hv_0$, the force is outward on electrons and ions, but it is not very much greater than normal because of the magnitude of the space charges and their velocities; there will be a voltage subtracted by space charge.

If we now terminate the container by a set of insulated slats, the moving ionized gas will terminate by precipitating upon these slats, if only the electric potential of the side plates is high enough (see Fig. 3).

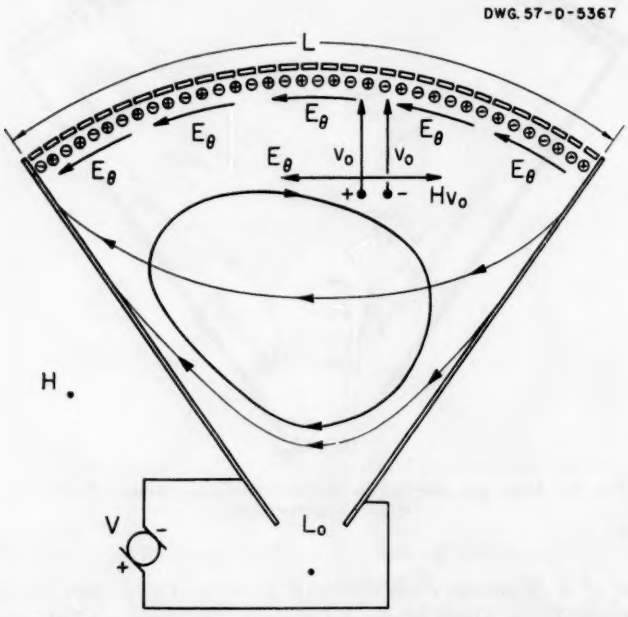


FIG. 3.—Ionic gas, expanding in actual radial container. Gas is of two isotopes, m_1 and m_2. Curved arrowed lines are current lines of negligible magnitude; closed curve is of considerable magnitude.

IV. *Electric Field and Potentials Associated with Magnetic Field.*—To calculate the electric fields and potentials, we need to know the value of v_0, since, with $E_\theta = Hv_0$, the mean motion of the ions and the accompanying electrons is uniformly outward. Of course, the random superimposed motion of the ions and electrons of

temperature T will give a current of both ions and electrons to the plates, but the currents will be equal to both plates.

If the initial radial velocity is v_{00} and the initial distance measured circumferentially between the side plates is L_0, then

$$E_\theta = Hv_{00}, \qquad V_0 = -L_0Hv_{00}, \tag{10}$$

is the open-circuit voltage between the plates.

Now we raise the voltage between the radial plates until it is more than enough to bring the discharge out to the insulated end plates. Now the field E_θ at the insulated plates will equal the negative current reaction, Hv_0. The potential of the last insulated plate will be $-LE_\theta$ greater than the potential of the first plate:

$$E_\theta = Hv_0; \qquad V = -LHv_0, \tag{11}$$

where L is the circumferential distance between the two side plates.

Since, for L/L_0 large, $S/S_0 = L/L_0$, then

$$\frac{V}{V_0} = \frac{L}{L_0}\frac{v_0}{v_{00}}. \tag{12}$$

V. *Gas Consists of Two Isotopic Ions: The Problem.*—Let the initial composition of the gas now include isotopic ions of two masses, m_1 and m_2; a radial velocity v_{00}; and a temperature T_0, giving a random velocity

$$\tfrac{1}{2}m_1\overline{v_1^2} = \tfrac{1}{2}m_2\overline{v_2^2} = \tfrac{1}{2}m_0\overline{u^2} = 1/2\ kT_0. \tag{13}$$

Now it is obvious that we cannot have simultaneous solutions of the *two* equations

$$\tfrac{1}{2}m_1v_{00}^2 + 3kT_0 = \tfrac{1}{2}m_1v_0'^2 + 3kT_0\left(\frac{v_{00}S_0}{v_0'S}\right)^{2/3} \tag{7'}$$

and

$$\tfrac{1}{2}m_2v_{00}^2 + 3kT_0 = \tfrac{1}{2}m_2v_0''^2 + 3kT_0\left(\frac{v_{00}S_0}{v_0''S}\right)^{2/3} \tag{7''}$$

for the same value, $v_0' = v_0''$, for the two *different mass* isotopes, m_1 and m_2.

If the gas were all of isotope m_1, the velocity v_0 would take on a value v_0', which satisfies equation (7') and

$$E_\theta' = Hv_0'; \qquad V' = -LHv_0', \tag{11'}$$

the first equation holding at all the end plates and the last equation holding at the last end plate.

If the gas were all of isotope m_2, the velocity v_0 would take on a value v_0'', which satisfies equation (7'') and

$$E_\theta'' = Hv_0''; \qquad V'' = -LHv_0'' \tag{11''}$$

for the field at all end plates and the potential at the last end plate.

If $m_1 < m_2$, then $v_0' > v_0''$, and $E_\theta' > E_\theta''$; $\quad -V' > -V''$.

If ions of mass m_1 and m_2 are both present, then the E_θ may be different for the different end plates, but we must have, for each one,

$$E_\theta' > E_\theta > E_\theta'', \qquad -V' > -V > -V_\theta''. \tag{11'''}$$

However, the temperature which the gas has at any end plate also would be different if the gas were made up entirely of isotope m_1 or of isotope m_2. If the gas were all isotope m_1, the temperature at each end plate would be

$$T' = T_0\left(\frac{v_{00}S_0}{v_0'S}\right)^{2/3} \tag{14'}$$

If the ions were all of isotope m_2, the temperature would be

$$T'' = T_0\left(\frac{v_{00}S_0}{v_0''S}\right)^{2/3}. \tag{14''}$$

The actual temperatures of the gas at the different end plates will all be different, but the temperatures T will satisfy

$$T' < T < T''. \tag{14'''}$$

VI. *Gas Consists of Two Isotopic Ions: Problem Solved.*—The problem set by the preceding section finds a ready solution if we remember that the assumption was made that the azimuthal composition of the ion stream remains constant for the one-ion-isotope case. For two isotopes, we do not need the azimuthal constancy of the ion stream, and it is readily seen that it is actually disturbed by the forces which set themselves up here.

E_θ' and E_θ'' and V' and V'' are respectively different for the case of the two ions taken separately. Letting w_1 and w_2 be the velocity of the ions perpendicular to the ion stream, we have

$$w_1 = 0 \qquad \text{when ions of } m_1 \text{ only are present;} \tag{15'}$$

$$w_2 = 0 \qquad \text{when ions of } m_2 \text{ only are present.} \tag{15''}$$

But each end plate can have but one electric potential, and it must take a value V, intermediate between V' and V'', that is,

$$-V' > -V > -V''. \tag{11'''}$$

The lighter ion, m_1, must be retarded radially by the interactions of the electrons which are affected by the other ion, m_2. The other, heavier ion, m_2, must be speeded up radially by the interactions of the electrons which are affected by the lighter ion, m_1.

The lighter ion, m_1, is then acted upon by an additional radial force coming from the additional interactions of electrons with ions m_2. This force is a radial force and is directed radially inward. Corresponding to this force, the velocity w_1 is directed to the left in the figures and is toward the negatively charged side plate.

The heavier ion, m_2, however, is acted upon by an additional radial force coming from the additional interaction of electrons with ions m_1. This force is a radial force and is directed radially outward. Corresponding to this force, the velocity w_2 is directed to the right in the figures and is toward the positively charged side plate.

The difference between w_1 and w_2 will be given by

$$H(w_1 - w_2) = -V' + V'' = H\,L(v_0' - v_0'') \tag{16}$$

or

$$w_1 - w_2 = L(v_0' - v_0'') = Lv_0 \frac{dm}{2m_1} \tag{17}$$

where $m_2 = m_1 + dm$.

The velocity w_1 varies from 0 next to the left-hand, negatively charged plate to $+Lv_0(dm/2m)$ next to the right-hand, positively charged plate; w_2 varies from $-Lv_0(dm/2m)$ next to the left-hand, negatively charged plate to 0 next to the right-hand, positively charged plate.

It is clear that these motions will make the end plate next to the negatively charged side plate have an excess of its share of the m_1, or lighter, ion, and will make the end plate next to the positively charged side plate have an excess of its share of the m_2, or heavier, ion.

It is also clear that there will be an opposing flow of ions due to the effect of the temperature upon the ions of the two masses. The excess of ions of mass m_1 in the end plates to the left will cause a thermal motion of ions of mass m_1 to the right. The excess of ions of mass m_2 in the end plates to the right will cause a thermal motion of ions of mass m_2 to the left. Fortunately, the temperature T falls off as $(v_{00}S_0/v_0S)^{\frac{2}{3}}$, as S increases.

Thus it is clear that the enrichment of the ions among the end plates varies per unit length nearly as the first power of the total length and that the enrichment over the whole length of end plates varies nearly as the square of the whole length of end plates.

VII. *Initial Conditions.*—A convenient source of ions and electrons is an electric arc of short length, $^1/_4$ inch, parallel to the magnetic field. The negative electrode is a cored carbon such as is manufactured by the National Carbon Company. It is hardly consumed at all. The positive electrode is of the metal whose isotopes are wanted. Some of the time we used solid uranium. Sometimes we used a mixture of uranium with uranium oxide.

With an arc of 10 amp., we obtained a positive-ion current of 500 ma. to 1 amp. We have tested an arc of 100 amp., obtaining a positive-ion current of 10 amp. No limitations of vacuum were encountered. With the metal anodes, the neutrals were much less than 10 per cent of the number of ions.

The radially disposed side plates should be placed as close as possible to the arc. We set the beginning of the side plates one inch from the arc. We adjusted the magnetic field so that when the upper and lower one-inch circles around the arc electrodes were excited by more than 20 volts, the radially disposed side plates with zero potential difference between them floated at 20 volts negative to the arc anode. About 6,000 gauss sufficed. The enrichment with -7 volts potential difference between the two radial plates, was with uranium, about one-half of 1 per cent.

The arc voltage was 20 to less than 100 volts. The voltage between the radial side plates would take negligible power. We obtained grams of uranium deposit per kilowatt of arc energy. The ion random energy was about 5 volts in each direction. $^1/_2m\overline{v_1^2} = {}^1/_2kT$. The mean ion energy was also about 5 volts. We obtained an ion current of about half an ampere. The vacuum vessel was about three feet high and four feet in diameter. However, we used about one-foot-diameter central portion.

GRAVITY ANOMALIES AT CONTINENTAL MARGINS

By J. Lamar Worzel and G. Lynn Shurbet

LAMONT GEOLOGICAL OBSERVATORY,* COLUMBIA UNIVERSITY, PALISADES, NEW YORK

Communicated by M. Ewing, April 27, 1955

Introduction.—The oceans are the normal part of the earth's crust and the continents the anomalous part, 64 per cent of the earth's surface being covered by oceans deeper than 1,000 fathoms. The *M*-discontinuity between the crust and the mantle lies 9–12 km. below sea level in ocean areas and 30–40 km. below sea level beneath the continents. At the continental margins the crustal thickness changes by a factor of about 5. There are few data on this transition. It is difficult to investigate by explosion seismology because of the thickness of the layers, the complex nature of the interfaces, layer slopes, changes of lithology, etc. Gravity surveys at present give the only data which cover this transition zone in detail.

Observations.—Gravity observations at 104 stations were made on board U.S.S. "Tusk" at the locations shown in Figure 1. The observations were made with a Meinesz pendulum apparatus.[1] The data for Browne corrections were obtained with an auxiliary long-period pendulum apparatus[2] loaned by Professor Vening

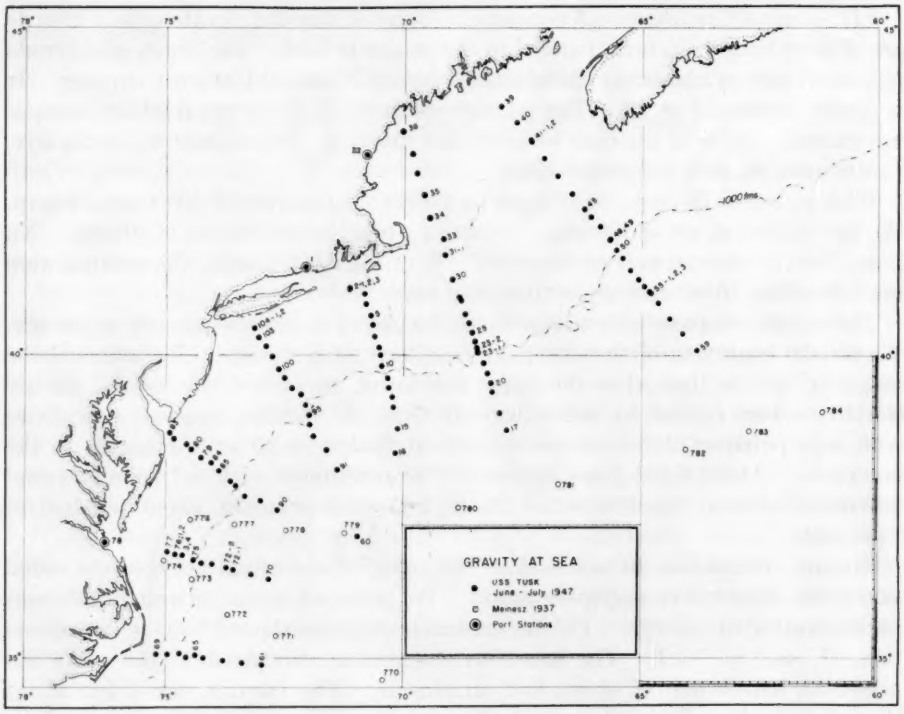


FIG. 1

Meinesz. Modifications in the apparatus were made and are described by Worzel and Ewing.[3]

Seven gravity sections are shown in Figures 2–8, the distance from a point near the shore line being given in nautical miles and kilometers. Part *A* of each figure shows the topographic section and the seismic data available. Part *B* shows the assumed structures and densities used for the gravity calculations. Part *C* shows a

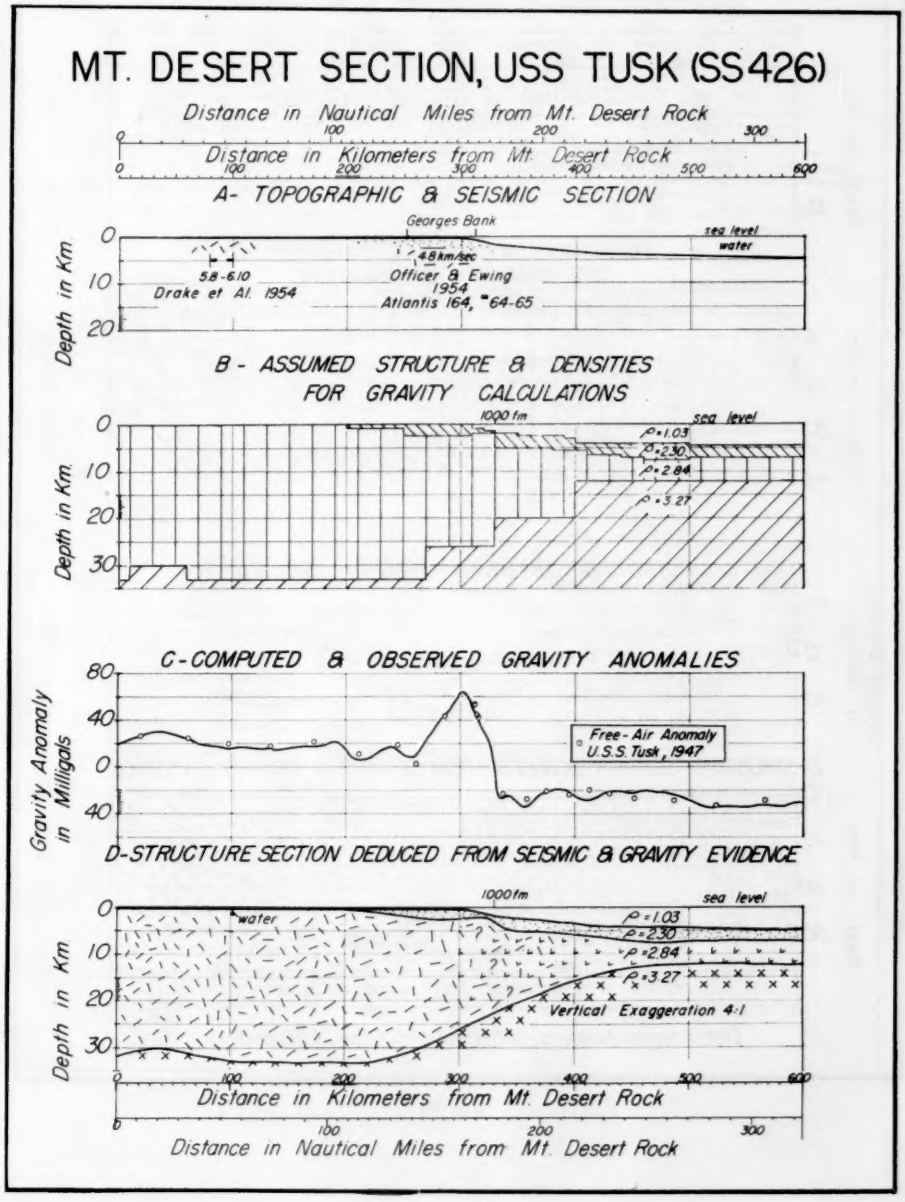


FIG. 2

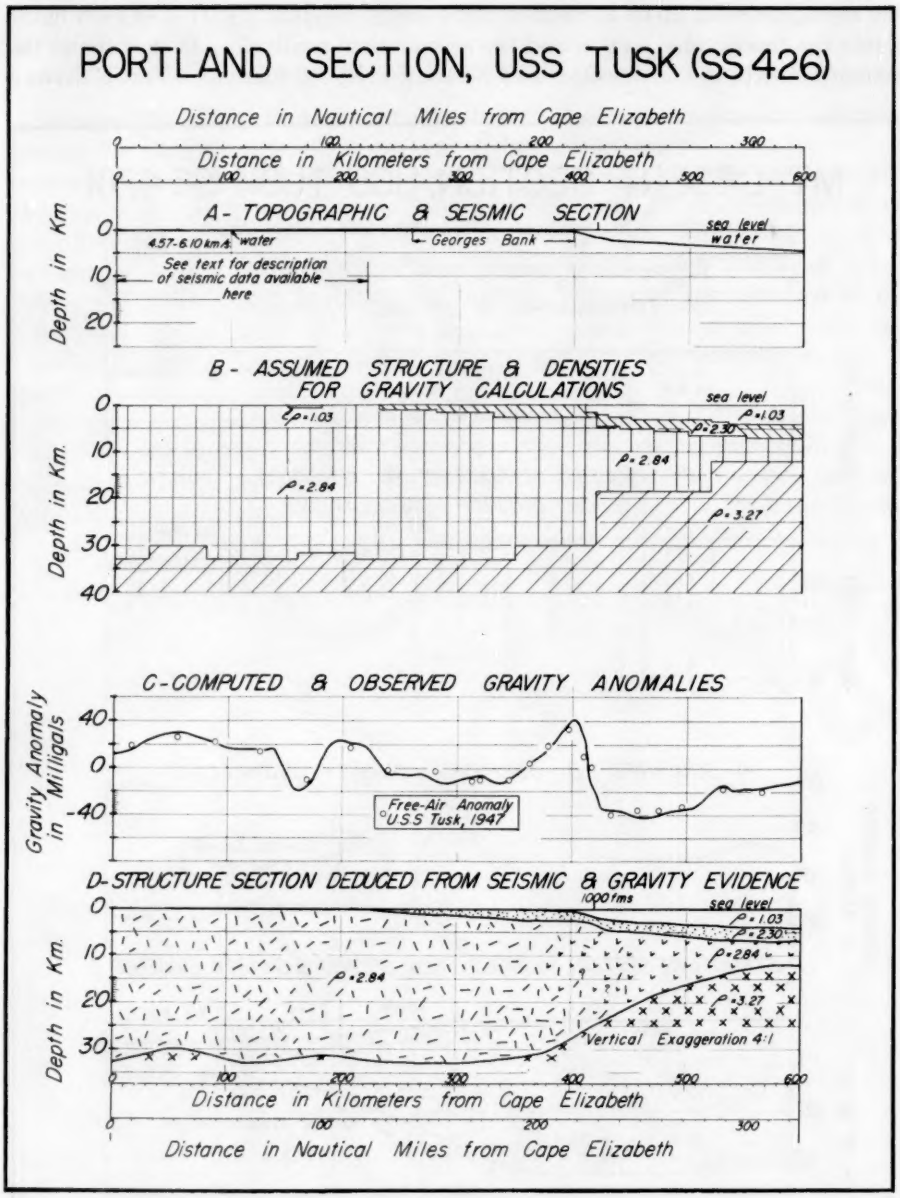

PORTLAND SECTION, USS TUSK (SS 426)
Distance in Nautical Miles from Cape Elizabeth
Distance in Kilometers from Cape Elizabeth
A - TOPOGRAPHIC & SEISMIC SECTION
Depth in Km
sea level
water
4.57-6.10 km/s water
Georges Bank
See text for description of seismic data available here
B - ASSUMED STRUCTURE & DENSITIES FOR GRAVITY CALCULATIONS
ρ=1.03
ρ=2.30
ρ=2.84
ρ=3.27
C - COMPUTED & OBSERVED GRAVITY ANOMALIES
Gravity Anomaly in Milligals
Free-Air Anomaly
U.S.S Tusk, 1947
D - STRUCTURE SECTION DEDUCED FROM SEISMIC & GRAVITY EVIDENCE
1000 fms
Vertical Exaggeration 4:1
Distance in Kilometers from Cape Elizabeth
Distance in Nautical Miles from Cape Elizabeth

FIG. 3

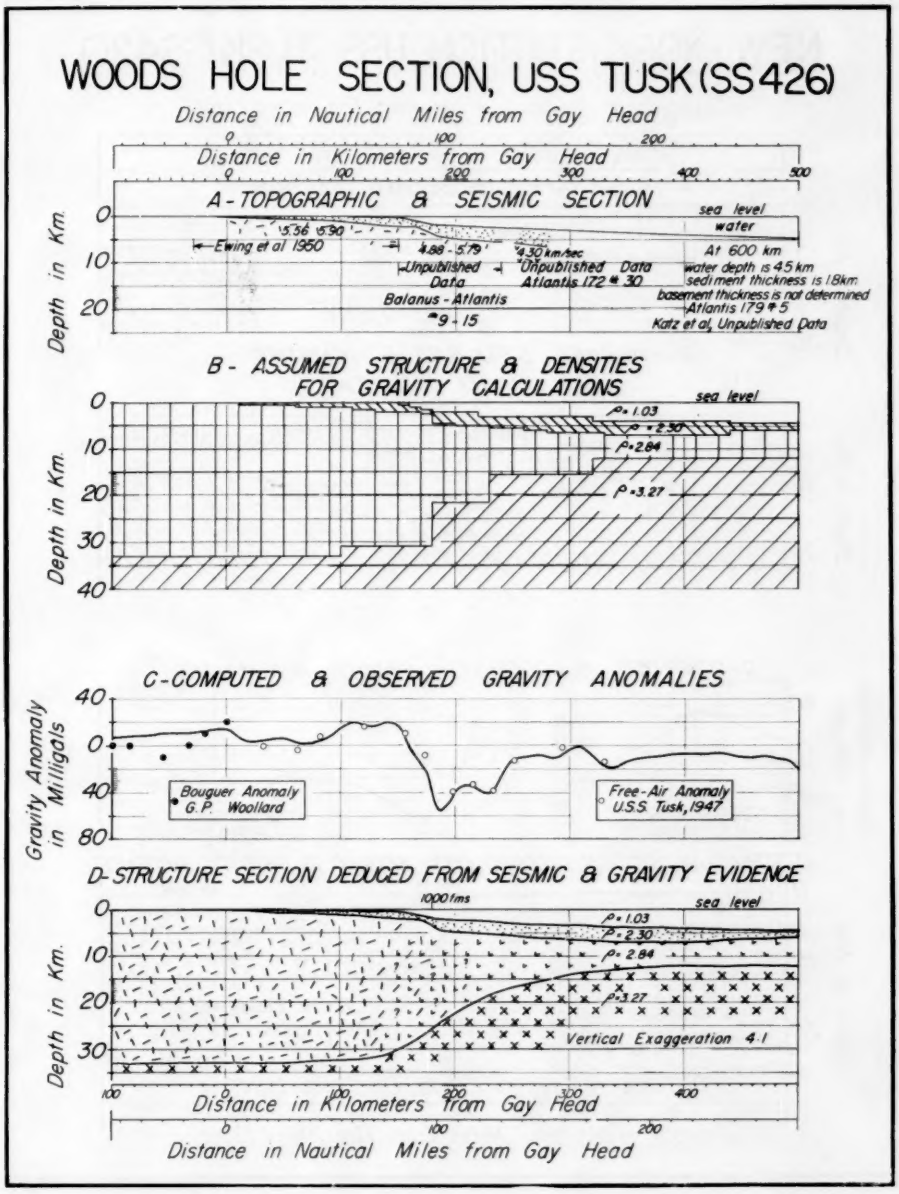

WOODS HOLE SECTION, USS TUSK (SS426)
Distance in Nautical Miles from Gay Head
Distance in Kilometers from Gay Head
A - TOPOGRAPHIC & SEISMIC SECTION
Depth in Km.
sea level
water
Ewing et al 1950
Unpublished Data
Balanus - Atlantis
Unpublished Data
Atlantis 172 # 30
At 600 km
water depth is 4.5 km
sediment thickness is 1.8km
basement thickness is not determined
Atlantis 179 # 5
Katz et al, Unpublished Data
B - ASSUMED STRUCTURE & DENSITIES FOR GRAVITY CALCULATIONS
ρ = 1.03
ρ = 2.30
ρ = 2.84
ρ = 3.27
C - COMPUTED & OBSERVED GRAVITY ANOMALIES
Gravity Anomaly in Milligals
Bouguer Anomaly G.P. Woollard
Free-Air Anomaly U.S.S. Tusk, 1947
D - STRUCTURE SECTION DEDUCED FROM SEISMIC & GRAVITY EVIDENCE
1000 fms
Vertical Exaggeration 4:1

FIG. 4

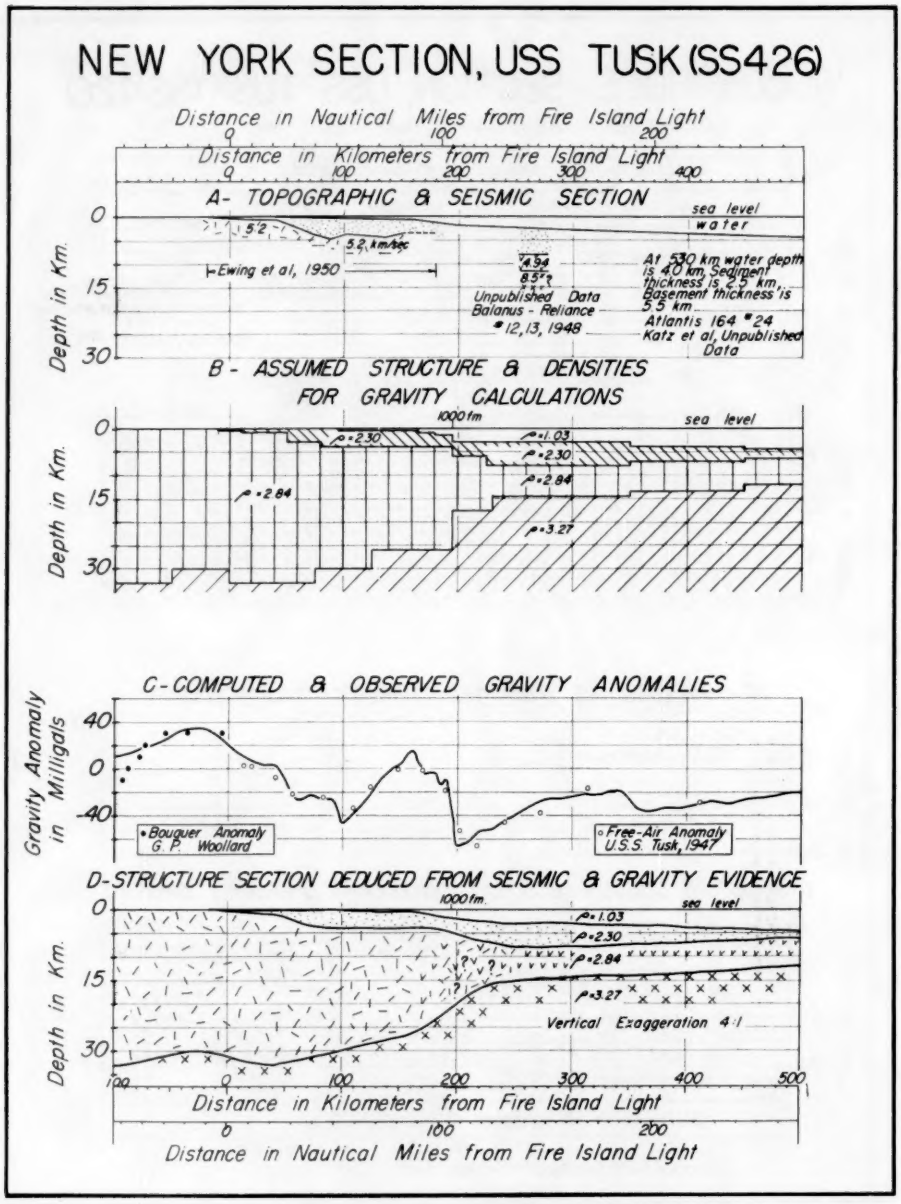

NEW YORK SECTION, USS TUSK (SS426)
Distance in Nautical Miles from Fire Island Light
Distance in Kilometers from Fire Island Light
A- TOPOGRAPHIC & SEISMIC SECTION
sea level
water
5.2
5.2 km/sec
Ewing et al, 1950
4.94
8.5?
Unpublished Data
Balanus - Reliance
12, 13, 1948
At 530 km water depth is 4.0 km, Sediment thickness is 2.5 km, Basement thickness is 5.5 km
Atlantis 164 #24
Katz et al, Unpublished Data
Depth in Km.
B - ASSUMED STRUCTURE & DENSITIES FOR GRAVITY CALCULATIONS
1000 fm
sea level
ρ=1.03
ρ=2.30
ρ=2.84
ρ=3.27
C - COMPUTED & OBSERVED GRAVITY ANOMALIES
Gravity Anomaly in Milligals
Bouguer Anomaly G. P. Woollard
Free-Air Anomaly U.S.S. Tusk, 1947
D - STRUCTURE SECTION DEDUCED FROM SEISMIC & GRAVITY EVIDENCE
Vertical Exaggeration 4:1

FIG. 5

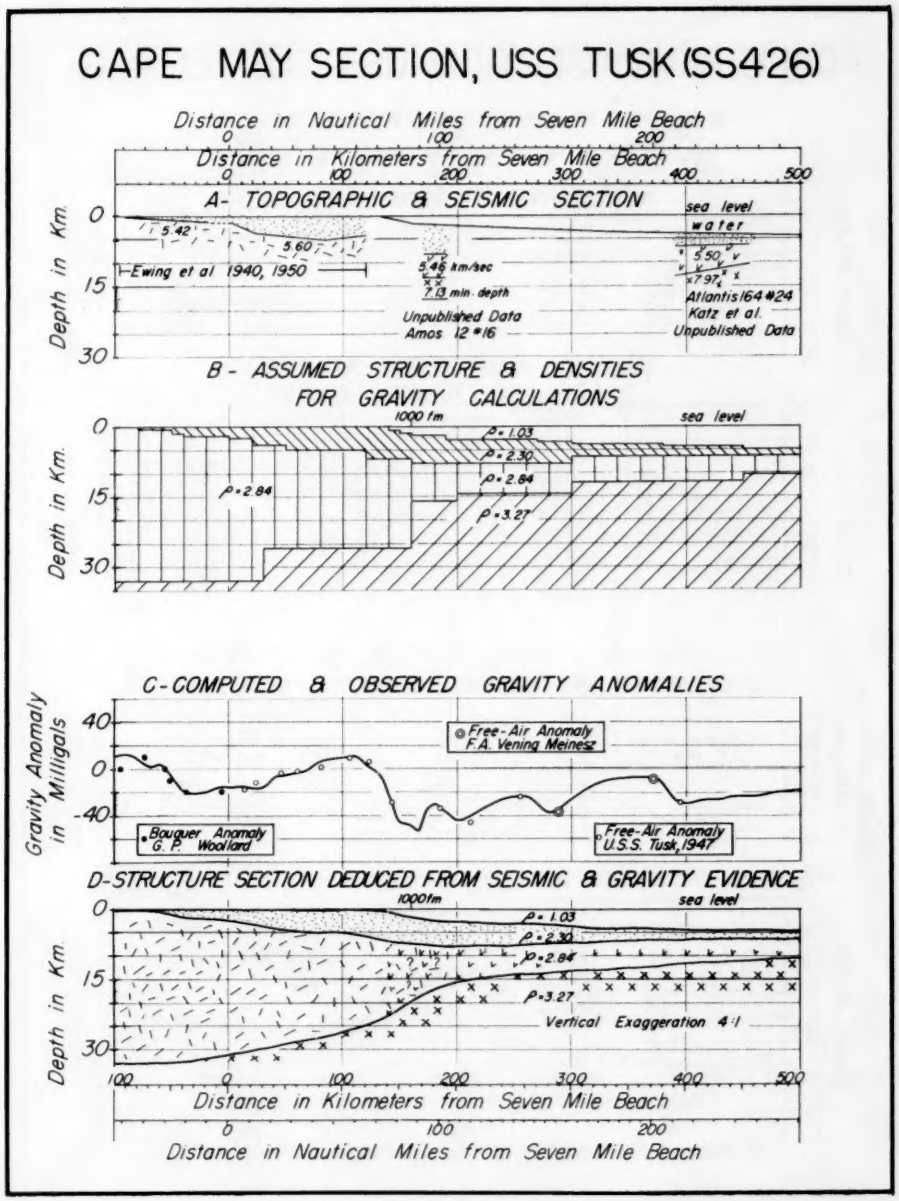

CAPE MAY SECTION, USS TUSK (SS426)
Distance in Nautical Miles from Seven Mile Beach
Distance in Kilometers from Seven Mile Beach
A- TOPOGRAPHIC & SEISMIC SECTION
sea level
water
Depth in Km.
5.42
5.60
Ewing et al 1940, 1950
5.46 km/sec
7.13 min. depth
Unpublished Data
Amos 12 #16
5.50
7.97
Atlantis 164 #24
Katz et al.
Unpublished Data
B - ASSUMED STRUCTURE & DENSITIES FOR GRAVITY CALCULATIONS
1000 fm
sea level
ρ = 1.03
ρ = 2.30
ρ = 2.84
ρ = 3.27
C - COMPUTED & OBSERVED GRAVITY ANOMALIES
Gravity Anomaly in Milligals
Free-Air Anomaly F.A. Vening Meinesz
Bouguer Anomaly G. P. Woollard
Free-Air Anomaly U.S.S. Tusk, 1947
D - STRUCTURE SECTION DEDUCED FROM SEISMIC & GRAVITY EVIDENCE
Vertical Exaggeration 4:1

FIG. 6

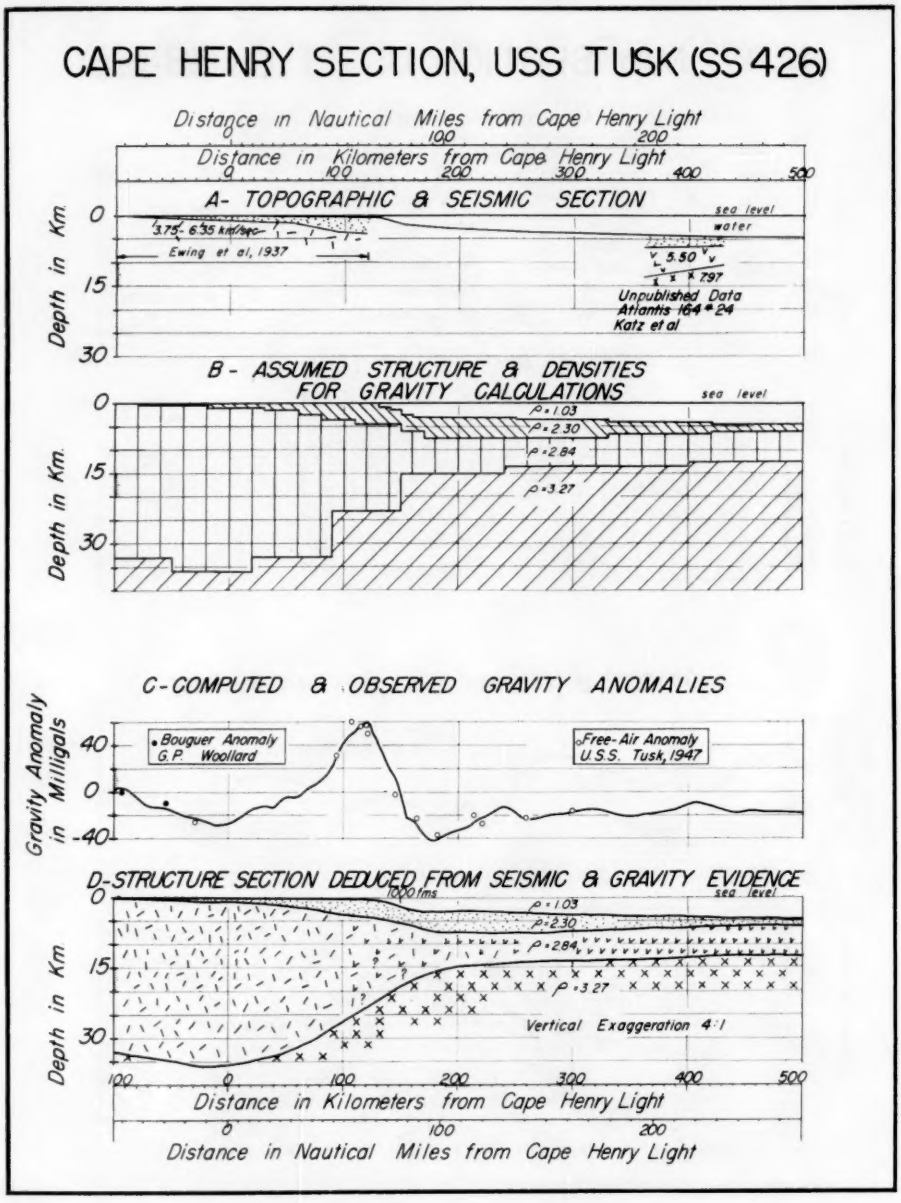

CAPE HENRY SECTION, USS TUSK (SS 426)
Distance in Nautical Miles from Cape Henry Light
Distance in Kilometers from Cape Henry Light
A- TOPOGRAPHIC & SEISMIC SECTION
Depth in Km
sea level
water
3.75- 6.35 km/sec
Ewing et al, 1937
5.50
7.97
Unpublished Data
Atlantis 164 & 24
Katz et al
B - ASSUMED STRUCTURE & DENSITIES FOR GRAVITY CALCULATIONS
ρ=1.03
ρ=2.30
ρ=2.84
ρ=3.27
C - COMPUTED & OBSERVED GRAVITY ANOMALIES
Gravity Anomaly in Milligals
Bouguer Anomaly G.P. Woollard
Free-Air Anomaly U.S.S. Tusk, 1947
D-STRUCTURE SECTION DEDUCED FROM SEISMIC & GRAVITY EVIDENCE
1000 fms
Vertical Exaggeration 4:1

FIG. 7

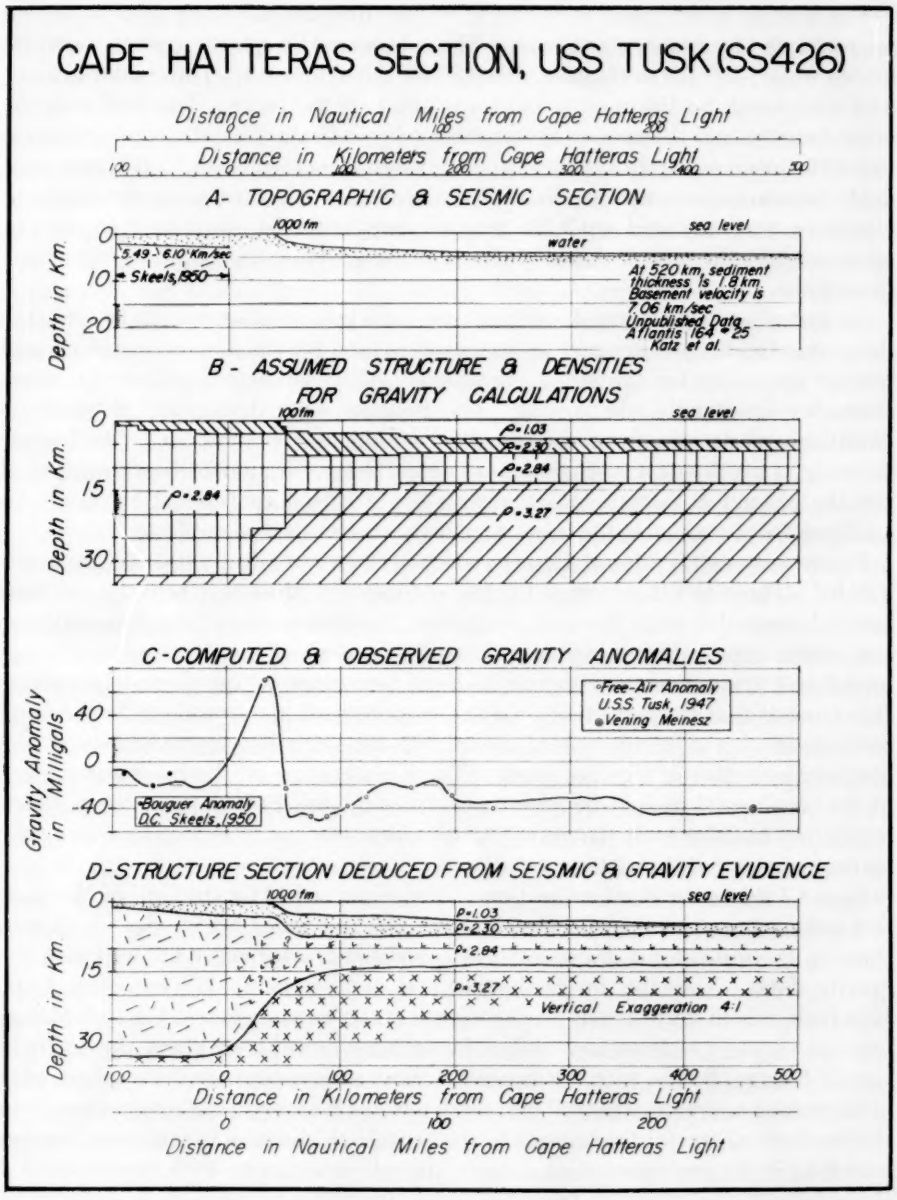

CAPE HATTERAS SECTION, USS TUSK (SS426)
Distance in Nautical Miles from Cape Hatteras Light
Distance in Kilometers from Cape Hatteras Light
A- TOPOGRAPHIC & SEISMIC SECTION
1000 fm
sea level
water
Depth in Km.
5.49 - 6.10 Km/sec
Skeels, 1950
At 520 km, sediment thickness is 1.8 km. Basement velocity is 7.06 km/sec
Unpublished Data Atlantis 164 - 25
Katz et al.
B - ASSUMED STRUCTURE & DENSITIES FOR GRAVITY CALCULATIONS
100 fm
sea level
ρ=1.03
ρ=2.30
ρ=2.84
ρ=3.27
C - COMPUTED & OBSERVED GRAVITY ANOMALIES
Gravity Anomaly in Milligals
Free-Air Anomaly U.S.S. Tusk, 1947
Vening Meinesz
Bouguer Anomaly
C.C. Skeels, 1950
D - STRUCTURE SECTION DEDUCED FROM SEISMIC & GRAVITY EVIDENCE
Vertical Exaggeration 4:1
Distance in Kilometers from Cape Hatteras Light
Distance in Nautical Miles from Cape Hatteras Light

FIG. 8

comparison of the anomaly curve computed from Part *B* and the observed free-air gravity anomalies. Part *D* shows the generalized structure deduced from the gravity and seismic evidence.

The gravity calculations were made by two-dimensional analysis of a section approximated by rectangular blocks. The values used for specific gravity were 1.03 for sea water, 3.27 for the mantle, and 2.30 for the sediments. The specific gravity 2.84 was chosen for the oceanic and continental crustal rocks. Together with the other densities and thicknesses chosen, this value indicates isostatic equilibrium between the mean continental column and the mean oceanic columns.[4] Where seismic depth determinations were available, they were adopted; otherwise, the depths of interfaces were adjusted until the gravity data could be adequately fitted. In sections where the seismic data were inadequate, guidance was obtained from nearby sections.

On several sections it was desirable to use land gravity observations to complete the gravity sections. Bouguer anomalies were used for the land observations and free-air anomalies for the sea observations. Although named differently, these anomalies are strictly comparable. The Bouguer anomaly on land depends on deviations of density from the standard continental column, and the free-air anomaly at sea depends on deviations of actual density from densities which would put the column in isostatic balance with the standard continental column. An additional contribution to the free-air anomaly arises from "edge effects."

Figure 2 shows the Mount Desert section. Only two seismic stations are available.[5, 6] There is little control for the sedimentary thickness beneath the continental rise and slope. However, in view of the other sections to be discussed later and unless improbably narrow and sharp fluctuations of the *M*-discontinuity are introduced, the sedimentary section deduced here must be approximately correct. It is obvious that the transition from the crust beneath the continents to the crust beneath the ocean occurs within about 200 km., starting approximately at the northern boundary of Georges Bank. There is structure on the basement surface at the continental margin beneath approximately the 1,000-fathom curve which is required in order to fit the gravity data, unless the line of section follows a submarine canyon in this vicinity.

Figure 3 shows the Portland section. The seismic data[5] for the Gulf of Maine in this section show less than 0.3 km. of sediments and water, which was too thin to show on the scale of our drawing. The computed gravity curve fits well with the gravity data. Note the dip in the curve at approximately 175 km. from Cape Elizabeth, which corresponds to the deeper water in the central Gulf of Maine. One can choose a sedimentary wedge thickening seaward from about the northern edge of Georges Banks, with the greatest thickness occurring near the southern edge of the section approximately 550 km. south of Cape Elizabeth. The transition from the continental crustal thicknesses to the oceanic thicknesses is somewhat steeper here than in the previous section. Again there is structure beneath the continental slope. Some fluctuation of the *M*-discontinuity must be allowed in order to fit the gravity data on the inner part of the Gulf of Maine. Any gradual change in the thickness of the sedimentary wedge could be compensated in gravity effect by a gradual shift in the *M*-discontinuity. The section chosen conforms with neighboring sections. The Gulf of Maine is a continental block that is flooded.

Figure 4 shows the Woods Hole section. Here we have seismic measurements[7] of the sedimentary thickness to the basement surface from shore to about 120 km. beyond the continental slope. Since we cannot move these boundaries, most of the adjustment for gravity calculations had to be done on the *M*-discontinuity. The fit of the computed curve to the observed points is satisfactory. G. P. Woollard (personal communication) provided the gravity values for the land part of the section. The northern end of the profile crosses the eastern end of a rather large gravity feature. We did not attempt to fit these data, as we had too little evidence of its structure within our section to do it justice. The transition of the *M*-discontinuity from land to sea is more rapid here than in the previous sections (which had little seismic control), although tapering off on the seaward end more gradually. The structure on the basement surfaces which we had to add in the previous sections is shown in the seismic data. The seismic data confirm that the greatest sedimentary thickness occurs on the oceanic side of the continental slope. This sedimentary wedge obviously thickens to about 350 km. from Gay Head and then thins seaward to a seismic station 100 km. beyond the end of the section. The bulk of the sediment lies beneath the continental rise.

Figure 5 shows the New York section. There is good seismic data[7] for the basement surface beneath the continental shelf. Beyond the continental slope there is one section at a distance of about 280 km. from Fire Island Light and a second seismic section 530 km. southeast of Fire Island Light just off the edge of the diagram. The fit of the computed curve with the observed data is quite satisfactory with the sections shown in section *B*. It was necessary to put some structure on the *M*-discontinuity surface beneath the continental part of the block in order to fit the curve adequately. Two great accumulations of sediment are found. The thickness is about 5 km. on the shelf and about 6 km. at 300 km. southeast of Fire Island Light. The rise of the *M*-discontinuity from typical depths starts beneath the first flexure of the basement, and typical oceanic crust is found at a distance of about 200 km. The greatest sedimentary thickness beneath the continental rise is found where the Hudson Canyon delta was reported by Ericson *et al.*[8]

Figure 6 shows the Cape May section. There is considerable seismic detail[7, 9] beneath the continental shelf, at the base of the continental slope, and seaward at about 420 km. The fit of the computed gravity curve with the observed data is quite satisfactory. There is less structure on the basement surface than for previous sections. The sediments thicken gradually seaward, achieve their greatest thickness beneath the continental slope, and then thin gradually seaward. A large part of this section cuts across the Hudson Canyon delta on the continental rise. The *M*-discontinuity rises more gradually beneath this section than the previous ones, starting its rise approximately at the shore line and achieving its typically oceanic depth at about 250 km. from the shore line. This is the greatest accumulation of sediments indicated on any of the sections.

Figure 7 shows the Cape Henry section. The seismic data[10] from the fall line to the 100-fathom curve is shown at the top. An additional seismic section 400 km. east of Cape Henry Light is available. The computed gravity curve fits quite satisfactorily with the observed values. Note the steepening of the gravity curve at about the 1,000-fathom curve. No gravity observations could be made in the interval from 0 to 90 km. from Cape Henry because of the water depth and the

very busy ship channel. The sediment thickens seaward again, reaching its greatest thickness at the foot of the continental slope or the top of the continental rise and then thins gradually across the continental rise. The *M*-discontinuity rises quite rapidly in this area from approximately 50 km. from Cape Henry Light to a normal sea depth at about 200 km. from Cape Henry Light.

Figure 8 shows the Cape Hatteras section. The data inshore is made available by Skeels.[11] There is no additional data until just beyond the end of this section at 520 km. from Cape Hatteras Light. The fit of the computed curve with the observed anomalies is quite satisfactory. Note how extremely steep the anomaly curve is near the 1,000-fathom curve. The region from 30 km. west to 30 km. east of Cape Hatteras Light was not observed because the water was too shallow. The sediments thicken seaward, achieving their maximum thickness at the base of the continental slope and then thins gradually seaward. Again the maximum thickness of sediments occurs beneath the continental rise. The *M*-discontinuity has the sharpest rise here of any of the sections, rising from its continental depth to a typically oceanic depth in a distance of only about 80 km.

Discussion and Conclusions.—We conclude that the true edge of this continent occurs at about the 1,000-fathom curve. The maximum sedimentary thickness is found near the base of the continental slope, and there is a significant amount of sediment across the whole continental rise. By far the greatest volume of sediments is found on the ocean side of the continental slope, probably accounting for the existence of the continental rise. This distribution of sediments probably results from the action of turbidity currents.

If all the sediment were removed and these areas remained closely in isostatic balance, as they are at present, there would be a continental shelf floored with basement rocks, a continental slope floored with basement rocks, no continental rise, and a deep oceanic area floored by nearly level basement rocks. This must represent an earlier stage in the development of this coast.

Sedimentation, mostly from the continent, must have produced the present structure. As sedimentation continues, the oceanic crust is depressed closely in isostatic equilibrium until the sediment surface reaches sea level. The continental slope moves seaward over this thickened sedimentary section as the top of the sediment approaches sea level. All these sections represent intermediate stages of this process, with the New York and Cape May sections the farthest advanced, probably owing to the much larger supply of sediments to this area. The Gulf Coast geosyncline represents a much more advanced stage of this process.[4] Of course, if orogeny occurs within the region, this process is interrupted.

The steepness of the *M*-discontinuity varies considerably and is not simply related to the near-surface structure at the continental margin. The continental crust thins fairly abruptly in about 200 km. to the oceanic crustal thicknesses.

Thanks are due to Commander Submarine Forces, Atlantic Fleet, and the officers and crew of U.S.S. "Tusk" (SS 426), Commander G. F. Gugliotta, commanding, for their support of the project.

Nelson C. Steenland, Paul C. Wuenschel, and Gordon R. Hamilton assisted in making the observations. Maurice Ewing supervised the work and provided advice and assistance. G. P. Woollard, S. Katz, John Ewing, and George Sutton made

data available to us in advance of publication. Fay Jones, Emily Hermann, Elizabeth S. Skinner, and Annette Trefzer assisted in reducing the data and preparing the manuscript.

This work was carried out under Contract N6-onr-271 Task Order 8 with the Office of Naval Research, Department of the Navy.

* This is Contribution No. 132 of the Lamont Geological Observatory of Columbia University.

[1] F. A. Vening Meinesz, *Theory and Practice of Pendulum Observations at Sea* (Delft: Technische Boekhandel en Drukkerij, J. Waltman, Jr., 1929).

[2] F. A. Vening Meinesz, *Theory and Practice of Pendulum Observations at Sea, Part II: Second Order Corrections, Terms of Browne and Miscellaneous Subjects* (Delft: Drukkerij, A. J. Waltmans 1941).

[3] J. Lamar Worzel and Maurice Ewing, *Trans. Am. Geophys. Union*, **31**, 917, 1950.

[4] J. Lamar Worzel and G. Lynn Shurbet, *Crust of the Earth* (Geological Society of America [in press]).

[5] Charles L. Drake, J. Lamar Worzel, and Walter C. Beckmann, *Bull. Geol. Soc. Amer.*, **65**, 957, 1954.

[6] C. B. Officer and Maurice Ewing, *Bull. Geol. Soc. Amer.*, **65**, 653, 1954.

[7] Maurice Ewing, J. L. Worzel, N. C. Steenland, and Frank Press, *Bull. Geol. Soc. Amer.*, **61**, 877, 1950.

[8] D. B. Ericson, Maurice Ewing, and Bruce C. Heezen, *Bull. Geol. Soc. Amer.*, **62**, 961, 1951.

[9] Maurice Ewing, George P. Woollard, and A. C. Vine, *Bull. Geol. Soc. Amer.*, **51**, 1821, 1940.

[10] Maurice Ewing, A. P. Crary, and H. M. Rutherford, *Bull. Geol. Soc. Amer.*, **48**, 753, 1937.

[11] D. C. Skeels, *Geophysics*, **15**, 413, 1950.

THE SEISMIC SURFACE PULSE

BY C. L. PEKERIS

DEPARTMENT OF APPLIED MATHEMATICS, WEIZMANN INSTITUTE, REHOVOT, ISRAEL

1. *Introduction.*—The problem under investigation is to determine the motion of the surface of a uniform elastic half-space produced by the application at the surface of a point pressure pulse varying with time like the Heaviside unit function. The original formulation of the problem is due to Lamb,[1] who synthesized the solution for the pulse from the periodic solution. Lamb's method is, however, very intricate. In a previous publication[2] the author gave an exact and closed expression for the vertical component of displacement for the case when the pressure pulse varies like the Heaviside unit function $H(t)$. The derivation of this result, as well as the solution for the horizontal displacement, are given in this paper. The seismic pulse problem was treated nearly simultaneously by Cagniard,[3] and more recently by Pinney[4] and Dix.[5] Because of the complexity of the analysis, it was thought worth while to reproduce in this and a subsequent publication the original solution for the surface source and the buried source.

2. *Formal Solution.*—In this section we derive a formal solution for the problem of the motion produced by a seismic source buried below the surface in a uniform elastic half-space, when the time variation of the pulse is $H(t)$. The solution for the surface source will then be obtained by letting the depth of source H approach zero. Referring to Figure 1, we choose a cylindrical system of co-ordinates with origin at the level of the source and the surface situated at $z = -H$. Quantities

referring to the portions of the medium below and above the source will be designated by the subscripts 1 and 2, respectively.

The space variation of the source will be specified by the condition that at the level of the source ($z = 0$) the surface integral of the applied stress ($p_{zz1} - p_{zz2}$) shall be equal to Z (negative):

$$2\pi\int_0^\infty (p_{zz1} - p_{zz2}) r\, dr = Z. \tag{1}$$

One representation of this point source is

$$(p_{zz1} - p_{zz2}) = \lim_{\epsilon \to 0} f(\epsilon, r),$$

$$f(\epsilon, r) = \frac{Z}{2\pi}\int_0^\infty J_0(\xi r) e^{-\epsilon\xi}\, \xi\, d\xi = \frac{Z\epsilon}{2\pi}(\epsilon^2 + r^2)^{-3/2} \tag{2}$$

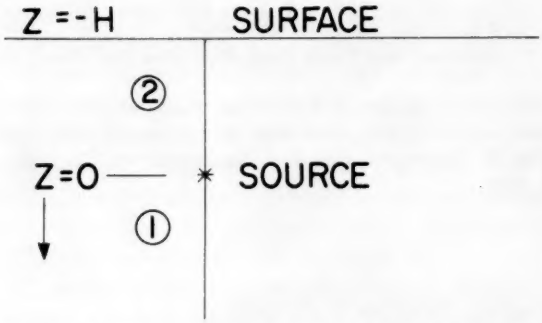


FIG. 1.—A point source is situated at a depth H below the surface of a uniform elastic half-space.

The vertical displacement w and the horizontal displacement q can be derived (Lamb,[1] p. 29) from

$$q = \phi_r + \chi_{rz},$$
$$w = \phi_z + \chi_{zz} - k^2\chi, \tag{3}$$

where the subscripts denote partial differentiation, and the potentials ϕ and χ satisfy the wave equations for dilatational and equivoluminal motion, respectively:

$$\nabla^2\phi - h^2\phi = 0, \qquad \nabla^2\chi - k^2\chi = 0, \tag{4}$$

$$h^2 = \frac{p^2}{c_p^2}, \qquad k^2 = \frac{p^2}{c^2}, \qquad c^2 = \frac{\mu}{\rho}, \qquad c_p^2 = \frac{\lambda + 2\mu}{\rho} = 3c^2. \tag{5}$$

Here c_p denotes the velocity of compressional waves, c the velocity of shear waves, and p denotes the operator $\partial/\partial t$. λ and μ are the elastic constants of the medium, which in the sequel we shall assume to be equal.

The normal stress p_{zz} and the shear stress p_{rz} are given by

$$p_{zz} = \lambda h^2\phi + 2\mu(\phi_{zz} + \chi_{zzz} - k^2\chi_z), \tag{6}$$

$$p_{rz} = \mu\left(\frac{\partial}{\partial r}\right)(2\phi_z + 2\chi_{zz} - k^2\chi). \tag{7}$$

It is understood that after we have obtained the operational representation of $w(r, z, p)$, say, the actual $w(r, z, t)$ will be obtained by performing the integration over the Bromwich contour

$$w(r, z, t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(\frac{e^{pt}}{p}\right) w(r, z, p)\, dp. \tag{8}$$

Appropriate solutions of equations (4) in regions 1 and 2 are

$$\phi_1 = Ae^{-k\alpha z}J_0(\xi r), \qquad \chi_1 = Be^{-k\beta z}J_0(\xi r), \tag{9}$$

$$\phi_2 = (Ce^{-k\alpha z} + De^{k\alpha z})J_0(\xi r), \qquad \chi_2 = (Ee^{-k\beta z} + Fe^{k\beta z})J_0(\xi r), \tag{10}$$

where

$$k\alpha = (\xi^2 + h^2)^{1/2}, \qquad k\beta = (\xi^2 + k^2)^{1/2}, \tag{11}$$

and the functions are to be integrated over a suitable path in the ξ-plane. This path, as well as the functions (of ξ and p) A, B, C, D, E, and F, are determined from the following six boundary conditions. At the level $z = 0$ the required continuity of q, w, and p_{rz} yields

$$(\phi + \chi_z)_1 = (\phi + \chi_z)_2, \tag{12}$$

$$(\phi_z + \chi_{zz} - k^2\chi)_1 = (\phi_z + \chi_{zz} - k^2\chi)_2, \tag{13}$$

$$(2\phi_z + 2\chi_{zz} - k^2\chi)_1 = (2\phi_z + 2\chi_{zz} - k^2\chi)_2, \tag{14}$$

while the source condition (1) is met, in the limit of vanishing ϵ, by putting

$$(p_{zz1} - p_{zz2}) = \left(\frac{Z}{2\pi}\right)J_0(\xi r)\xi \tag{15}$$

and integrating with respect to ξ from zero to infinity. At the surface $z = -H$, p_{zz} and p_{rz}, as given by equations (6) and (7), must vanish. Solving these six boundary equations for A, B, C, D, E, and F, we get

$$F = \frac{Z\xi}{12\pi\mu k\beta h^2}, \qquad D = -k\beta F, \qquad B = E + F, \qquad A = C - D, \tag{16}$$

$$e^{k\alpha H}C = \frac{k\beta F}{M}\{[(2\xi^2 + k^2)^2 + 4k^2\xi^2\alpha\beta]e^{-k\alpha H} - 4\xi^2(2\xi^2 + k^2)e^{-k\beta H}\}, \tag{17}$$

$$e^{k\beta H}E = \frac{F}{M}\{4k^2(2\xi^2 + k^2)\alpha\beta e^{-k\alpha H} - [(2\xi^2 + k^2)^2 + 4k^2\xi^2\alpha\beta]e^{-k\beta H}\}, \tag{18}$$

$$M \equiv [(2\xi^2 + k^2)^2 - 4k^2\xi^2\alpha\beta]. \tag{19}$$

Substituting these values into formulas (9) and (10), we can derive from equations (3) the expressions for the operational representation of the displacements everywhere in the medium. The values of the latter at the surface are

$$w(p) \equiv w(r, 0, p) = \frac{Zk}{2\pi\mu}\int_0^\infty J_0(\xi r)\xi[-(2\xi^2 + k^2)e^{-k\alpha H} + 2\xi^2 e^{-k\beta H}]\left(\frac{\alpha}{M}\right)d\xi, \tag{20}$$

$$q(p) \equiv q(r, 0, p) = -\frac{Z}{2\pi\mu}\int_0^\infty J_1(\xi r)\xi^2[-2k^2\alpha\beta e^{-k\alpha H} + (2\xi^2 + k^2)e^{-k\beta H}] \times \left(\frac{1}{M}\right)d\xi. \tag{21}$$

3. *Vertical Displacement.*—The expression for the operational representation of the vertical displacement in the case of a surface source can be obtained from formula (20) by letting the depth of source H approach zero:

$$w(p) = \frac{Zk^3}{2\pi\mu}\int_0^\infty J_0(\xi r)\xi\alpha[(2\xi^2 + k^2)^2 - 4k^2\xi^2\alpha\beta]^{-1}\,d\xi. \tag{22}$$

By writing $\xi = kx$, this expression simplifies to

$$w(p) = \left(\frac{Zk}{2\pi\mu}\right)N(kr), \tag{23}$$

$$N = \int_0^\infty J_0(krx)xm(x)\,dx, \tag{24}$$

where

$$m(x) = \frac{\alpha}{[(2x^2 + 1)^2 - 4x^2\alpha\beta)]'}$$

$$\alpha = \sqrt{x^2 + 1/3}, \qquad \beta = \sqrt{x^2 + 1}, \qquad k = \frac{p}{c}, \tag{25}$$

and k appears only in the external factor and in $J_0(krx)$.

The interpretation of the external factor is simply $\partial/\partial t$. In order to interpret the integral, we transform it into the form $\int \exp\,[-kf(r, x)]g(x)\,dx$, *with* $f(r, x)$ *real.* The interpretation of the latter is $\int g(x)H[t - (1/c)f(r, x)]\,dx$, H denoting the Heaviside unit function. This can be accomplished by transforming the path of integration in equation (24) into one along the imaginary axis v in the complex x-plane, when $J_0(krx)$ goes over into $K_0(krv)$, the interpretation of which follows from the representation

$$K_0(krv) = \int_0^\infty \exp\,(-krv\cosh\theta)\,d\theta. \tag{26}$$

Now $m(x)$ has branch points at $x = \pm(i/\sqrt{3})$, $\pm i$, and poles at

$$x = \pm i\gamma, \qquad \gamma = 1/2\sqrt{3 + \sqrt{3}}. \tag{27}$$

We shall cut up the x-plane in such a manner as to leave $m(x)$ regular to the right of the imaginary axis, in order to be able to apply to the integral in ecuation (24) the theorem of Bateman and Pekeris,[6] leading to

$$N = \int_0^\infty J_0(krx)xm(x)\,dx = -\left(\frac{2}{\pi}\right)I\int_0^\infty K_0(krv)vm(iv)\,dv, \tag{28}$$

I denoting the imaginary part. This is shown in Figure 2, where the values of α and β in the various portions of the imaginary axis are also indicated. Near the pole $i\gamma$ the path is indented by a semicircle passed in the positive direction. It is easy to see that the integral over the semicircle is real and therefore does not contribute to equation (28). We must, however, take the principal value of the integral in equation (28) once we pass the pole. Since, moreover, $Im(iv) = 0$ for $v < 1/\sqrt{3}$, we obtain

$$N = -\left(\frac{2}{\pi}\right) IP \int_{1/\sqrt{3}}^{\infty} K_0(krv)vm(iv)\, dv, \tag{29}$$

where P denotes the principal value.

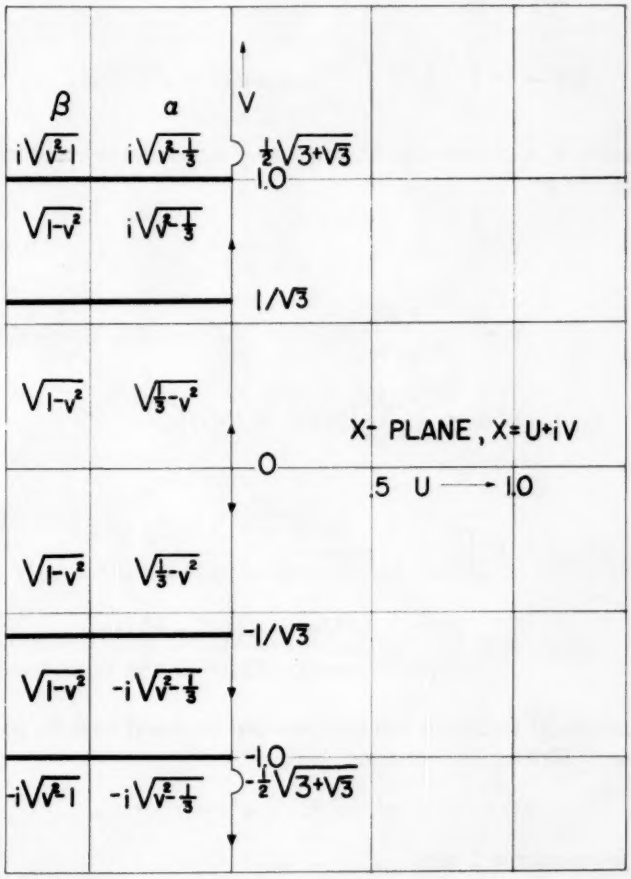


FIG. 2.—The complex x-plane is cut up by branch lines (thick) at the branch points $v = \pm(1/\sqrt{3})$ and $v = \pm 1.0$. The poles are situated at $v = \pm\frac{1}{2}\sqrt{3+\sqrt{3}}$. α and β denote the values of $\sqrt{\frac{1}{3}+x^2}$ and of $\sqrt{1+x^2}$ on the various portions of the v-axis.

Now the interpretation of $K_0(krv)$ is

$$K_0(krv) = \int_0^{\infty} \exp\left(\frac{-rpv\cosh\theta}{c}\right) d\theta \rightarrow \qquad 0,\ v > \tau,$$

$$\rightarrow \cosh^{-1}\left(\frac{\tau}{v}\right) \qquad v < \tau, \tag{30}$$

where $\tau = (ct/r)$, and $\rightarrow$ denotes operational interpretation. Hence

$$kK_0(krv) \rightarrow \frac{1}{c}\left(\frac{\partial}{\partial t}\right)\cosh^{-1}\left(\frac{\tau}{v}\right) = \frac{1}{r}(\tau^2 - v^2)^{-1/2}, \qquad v < \tau,$$

$$= 0, \qquad v > \tau, \quad (31)$$

$$kN \rightarrow 0, \qquad \tau < \frac{1}{\sqrt{3}},$$

$$kN \rightarrow -\left(\frac{2}{\pi r}\right) IP \int_{1/\sqrt{3}}^{\tau} vm(iv)(\tau^2 - v^2)^{-1/2}\, dv, \qquad \tau > \frac{1}{\sqrt{3}}. \quad (32)$$

Using the values of α, β shown in Figure 2 to evaluate $Im(iv)$, and substituting in equation (23), we get

$$w(\tau) = 0, \qquad \tau < \frac{1}{\sqrt{3}},$$

$$w(\tau) = \left(\frac{3Z}{\pi^2 \mu r}\right) G_1(\tau), \qquad \frac{1}{\sqrt{3}} < \tau < 1, \quad (33)$$

$$w(\tau) = \left(\frac{3Z}{\pi^2 \mu r}\right)[G_1(\tau) + G_2(\tau)], \qquad \tau > 1, \quad (34)$$

where

$$G_1(\tau) = P \int_{1/\sqrt{3}}^{\tau} \frac{v\sqrt{v^2 - 1/3}(1 - 2v^2)^2\, dv}{\sqrt{\tau^2 - v^2}(3 - 24v^2 + 56v^4 - 32v^6)}, \quad (35)$$

$$G_2(\tau) = P \int_{1}^{\tau} \frac{v^3\sqrt{v^2 - 1}[4v^2 - 4/3]\, dv}{\sqrt{\tau^2 - v^2}(3 - 24v^2 + 56v^4 - 32v^6)}. \quad (36)$$

The integrals $G_1(\tau)$ and $G_2(\tau)$ can be evaluated in closed form by partial fraction decomposition. Writing, in equation (35),

$$v^2 = 1/3 + \omega^2 \sin^2\theta; \qquad \omega^2 = \tau^2 - 1/3, \quad (37)$$

the integral is transformed into

$$G_1(\tau) = \frac{1}{96} P \int_0^{\pi/2} d\theta \left\{-12 + \frac{1}{[1/12 + \omega^2 \sin^2\theta]} - \frac{B}{(-b + \omega^2 \sin^2\theta)} - \frac{C}{(c + \omega^2 \sin^2\theta)}\right\}, \quad (38)$$

$$B = 3 + \frac{5}{\sqrt{3}}, \qquad b = 5/12 + \frac{\sqrt{3}}{4}, \qquad C = 3 - \frac{5}{\sqrt{3}}, \qquad c = \frac{\sqrt{3}}{4} - 5/12. \quad (39)$$

Using the result

$$\int_0^{\pi/2} d\theta(\alpha^2 + \omega^2 \sin^2\theta)^{-1} = \frac{\pi}{2\alpha}(\alpha^2 + \omega^2)^{-1/2}, \quad (40)$$

$$P\int_0^{\pi/2} d\theta(-\beta^2+\omega^2\sin^2\theta)^{-1} = 0, \qquad \beta<\omega$$

$$= -\frac{\pi}{2\beta}(\beta^2-\omega^2)^{-1/2}, \qquad \beta>\omega, \quad (41)$$

we obtain

$$G_1(\tau) = \frac{\pi}{96}\times$$

$$\left\{-6+\frac{\sqrt{3}}{\sqrt{\tau^2-1/4}}+\frac{\sqrt{3\sqrt{3}+5}}{\sqrt{1/4(3+\sqrt{3})-\tau^2}}-\frac{\sqrt{3\sqrt{3}-5}}{\sqrt{2\tau-1/4(3-\sqrt{3})}}\right\}, \qquad \tau<\gamma,$$

$$G_1(\tau) = \frac{\pi}{96}\left\{-6+\frac{\sqrt{3}}{\sqrt{\tau^2-1/4}}-\frac{\sqrt{3\sqrt{3}-5}}{\sqrt{\tau^2-1/4(3-\sqrt{3})}}\right\}, \qquad \tau>\gamma. \quad (42)$$

Similarly, using the substitution

$$v^2 = 1+\tilde{\omega}^2\sin^2\theta, \qquad \tilde{\omega}^2 = \tau^2-1, \quad (43)$$

in equation (36), we obtain

$$G_2(\tau) = 1/24\int_0^{\pi/2} d\theta[-3-3\cdot(3+4\tilde{\omega}^2\sin^2\theta)^{-1}-(1+\sqrt{3})\cdot$$
$$(1-\sqrt{3}+4\tilde{\omega}^2\sin^2\theta)^{-1}-(1-\sqrt{3})(1+\sqrt{3}+4\tilde{\omega}^2\sin^2\theta)^{-1}]$$

$$= \frac{\pi}{96}\left\{-6-\frac{\sqrt{3}}{\sqrt{\tau^2-1/4}}+\frac{\sqrt{3\sqrt{3}+5}}{\sqrt{1/4(3+\sqrt{3})-\tau^2}}+\right.$$

$$\left.\frac{\sqrt{3\sqrt{3}-5}}{\sqrt{\tau^2-1/4(3-\sqrt{3})}}\right\}, \qquad \tau<\gamma,$$

$$= \frac{\pi}{96}\left\{-6-\frac{\sqrt{3}}{\sqrt{\tau^2-1/4}}+\frac{\sqrt{3\sqrt{3}-5}}{\sqrt{\tau^2-1/4(3-\sqrt{3})}}\right\}, \qquad \tau>\gamma. \quad (44)$$

It follows that

$$w(\tau) = 0 \qquad \tau<\frac{1}{\sqrt{3}}, \quad (45)$$

$$w(\tau) = -\frac{Z}{32\pi\mu r}\left\{6-\frac{\sqrt{3}}{\sqrt{\tau^2-\frac{1}{4}}}-\frac{\sqrt{3\sqrt{3}+5}}{\sqrt{\frac{3}{4}+\frac{\sqrt{3}}{4}-\tau^2}}+\frac{\sqrt{3\sqrt{3}-5}}{\sqrt{\tau^2+\frac{\sqrt{3}}{4}-\frac{3}{4}}}\right\},$$

$$\frac{1}{\sqrt{3}}<\tau<1, \quad (46)$$

$$w(\tau) = -\frac{Z}{16\pi\mu r}\left\{6 - \frac{\sqrt{3\sqrt{3}+5}}{\sqrt{\frac{3}{4}+\frac{\sqrt{3}}{4}-\tau^2}}\right\}, \quad 1 < \tau < \gamma = {}^1/_2\sqrt{3+\sqrt{3}}, \quad (47)$$

$$w(\tau) = -\frac{3Z}{8\pi\mu r}, \quad \tau > \gamma. \quad (48)$$

4. *The Horizontal Displacement.*—The operational representation $q(p)$ for the horizontal displacement at the surface in case of a surface source can be obtained from formula (21) by putting $H = 0$:

$$q(p) = -\frac{Z}{2\pi\mu}\int_0^\infty \frac{J_1(\xi r)\xi^2[(2\xi^2+k^2)-2k^2\alpha\beta]\,d\xi}{[(2\xi^2+k^2)^2-4k^2\xi^2\alpha\beta]}, \quad (49)$$

which simplifies again, by the substitution $\xi = kx$, to

$$q(p) = \frac{Z}{2\pi\mu}\left(\frac{\partial Q}{\partial r}\right), \quad (50)$$

$$Q = \int_0^\infty J_0(krx)n(x)x\,dx, \quad (51)$$

$$n(x) = \frac{(2x^2+1)-2\sqrt{x^2+1}\,\sqrt{x^2+{}^1/_3}}{(2x^2+1)^2-4x^2\sqrt{x^2+1}\,\sqrt{x^2+{}^1/_3}}. \quad (52)$$

By cutting up the complex x-plane in the manner shown in Figure 2, we may again apply the Bateman-Pekeris theorem to equation (51), leading to

$$Q = \int_0^\infty J_0(krx)n(x)x\,dx = -\left(\frac{2}{\pi}\right)I\int_0^\infty K_0(krv)n(iv)v\,dv - {}^1/_4K_0(k\gamma), \quad (53)$$

where the last term arises from the residues at the poles $x = \pm i\gamma$. Now

$$\begin{aligned} In(iv) &= 0, && v < \frac{1}{\sqrt{3}} \\ &= \frac{\sqrt{v^2-{}^1/_3}\,\sqrt{1-v^2}(12v^2-6)}{(3-24v^2+56v^4-32v^6)}, && \frac{1}{\sqrt{3}} < v < 1, \quad (54) \\ &= 0, && v > 1, \end{aligned}$$

so that

$$Q = \frac{1}{\pi}\int_{1/\sqrt{3}}^1 \frac{K_0(krv)\sqrt{v^2-{}^1/_3}\,\sqrt{1-v^2}(12-24v^2)v\,dv}{(3-24v^2+56v^4-32v^6)} - \frac{K_0(k\gamma)}{4}. \quad (55)$$

In order to obtain the interpretation of $\partial Q/\partial r$, we differentiate equation (30) with respect to r and get

$$\begin{aligned} \left(\frac{\partial}{\partial r}\right)K_0(krv) &\to 0, && v > \tau, \\ &\to -\frac{\tau}{r\sqrt{\tau^2-v^2}}, && v < \tau, \quad (56) \end{aligned}$$

which, when substituted into equation (50), yields

$$\begin{aligned} q(\tau) &= 0, && \tau < \frac{1}{\sqrt{3}}, \\ q(\tau) &= -\left(\frac{Z}{2\pi^2\mu r}\right)\tau R_1(\tau), && \frac{1}{\sqrt{3}} < \tau < 1, \\ q(\tau) &= -\left(\frac{Z}{2\pi^2\mu r}\right)\tau R_2(\tau), && 1 < \tau < \gamma, \quad (57) \\ q(\tau) &= -\left(\frac{Z}{2\pi^2\mu r}\right)\tau R_2(\tau) + \frac{Z}{8\pi\mu r}(\tau^2 - \gamma^2)^{-1/2}, && \tau > \gamma, \end{aligned}$$

where

$$R_1(\tau) = \int_{1/\sqrt{3}}^{\tau} \frac{v\sqrt{v^2 - 1/3}\,\sqrt{1 - v^2}(12 - 24v^2)\,dv}{\sqrt{\tau^2 - v^2}(3 - 24v^2 + 56v^4 - 32v^6)}, \quad (58)$$

$$R_2(\tau) = \int_{1/\sqrt{3}}^{1} \frac{v\sqrt{v^2 - 1/3}\,\sqrt{1 - v^2}(12 - 24v^2)\,dv}{\sqrt{\tau^2 - v^2}(3 - 24v^2 + 56v^4 - 32v^6)}. \quad (59)$$

By partial fraction decomposition, $R_1(\tau)$ and $R_2(\tau)$ can be expressed in terms of elliptic integrals, as follows: Using the substitution (37) in equation (58), $R_1(\tau)$ is transformed into

$$\begin{aligned} R_1(\tau) &= -\frac{1}{8}\sqrt{\frac{3}{2}}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}} \left\{6 - \frac{18}{(1 + 8k^2\sin^2\theta)} + \right. \\ &\left. \frac{(6 - 4\sqrt{3})}{[1 - (12\sqrt{3} - 20)k^2\sin^2\theta]} + \frac{(6 + 4\sqrt{3})}{[1 + (12\sqrt{3} + 20)k^2\sin^2\theta]}\right\} \\ &= -\frac{1}{8}\sqrt{\frac{3}{2}}\{6K(k) - 18\Pi(8k^2, k) + (6 - 4\sqrt{3})\Pi[-(12\sqrt{3} - 20)k^2, k] + \\ &\qquad (6 + 4\sqrt{3})\Pi[(12\sqrt{3} + 20)k^2, k)\}, \quad (60) \end{aligned}$$

where

$$k^2 = 1/2(3\tau^2 - 1), \quad (61)$$

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}, \qquad \Pi(n, k) = \int_0^{\pi/2} \frac{d\theta}{(1 + n\sin^2\theta)\sqrt{1 - k^2\sin^2\theta}}. \quad (62)$$

$\Pi(n, k)$ can be expressed in terms of incomplete elliptic integrals,[7] for which tables are available. Similarly, the substitution

$$v^2 = \frac{1 + 2\sin^2\theta}{3} \quad (63)$$

transforms the integral in equation (59) into

$$R_2(\tau) = -\frac{1}{4\omega}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-\kappa^2\sin^2\theta}}\left\{3-\frac{9}{(8\sin^2\theta+1)}+\frac{(3+\sqrt{3})}{(8\sin^2\theta-5-3\sqrt{3})}+\frac{(3-\sqrt{3})}{(8\sin^2\theta-5+3\sqrt{3})}\right\}$$

$$= -\frac{1}{4\omega}\left\{3K(\kappa)-9\Pi(8,\kappa)-(2\sqrt{3}-3)\Pi[-(12\sqrt{3}-20),\kappa]+(2\sqrt{3}+3)\Pi[(12\sqrt{3}+20),\kappa]\right\}, \quad (64)$$

$$\kappa^2 = \frac{2}{3\tau^2-1}. \quad (65)$$

Substituting these into equations (57), we finally get

$$q(\tau) = 0, \qquad \tau < \frac{1}{\sqrt{3}},$$

$$q(\tau) = \frac{\sqrt{3}/2\tau Z}{16\pi^2\mu r}\left\{6K(k)-18\Pi(8k^2,k)+(6-4\sqrt{3})\Pi[-(12\sqrt{3}-20)k^2,k]+(6+4\sqrt{3})\Pi(12\sqrt{3}+20)k^2,k]\right\}, \qquad 1<\tau<\frac{1}{\sqrt{3}},$$

$$q(\tau) = \frac{\sqrt{3}/2\tau\kappa Z}{16\pi^2\mu r}\left\{6K(\kappa)-18\Pi(8,\kappa)+(6-4\sqrt{3})\Pi[-(12\sqrt{3}-20),\kappa]+(6+4\sqrt{3})\Pi[(12\sqrt{3}+20),\kappa]\right\}, \qquad 1<\tau<\gamma=\tfrac{1}{2}\sqrt{3+\sqrt{3}},$$

$$q(\tau) = \frac{\sqrt{3}/2\tau\kappa Z}{16\pi^2\mu r}\left\{6K(\kappa)-18\Pi(8,\kappa)+(6-4\sqrt{3})\Pi[-(12\sqrt{3}-20),\kappa]+(6+4\sqrt{3})\Pi[(12\sqrt{3}+20),\kappa]\right\}+\frac{Z\tau}{8\pi\mu r\sqrt{\tau^2-\gamma^2}}, \qquad \tau>\gamma \quad (66)$$

5. *Discussion of Results.*—The vertical displacement at the surface $w(\tau)$ due to the application of a point surface pressure pulse of the form $H(t)$ was computed from equations (45)–(48), and is plotted in Figure 3. The applied pressure being downward, the steady-state vertical displacement is downward. The initial displacement, however, at the time of the arrival of the P-wave, is upward. We note that the shear wave is marked only by a discontinuity in slope of $w(\tau)$. The displacement becomes infinite at the time of the arrival of the Rayleigh wave, and subsequently it reverts to the steady-state value.

Similarly, the horizontal displacement $q(\tau)$ was computed from equations (66), and is plotted in Figure 4. The ultimate displacement is inward, but the initial displacement at the P-epoch is outward. The arrival of the shear wave is marked only by a change of slope of $q(\tau)$, which is even less marked than in $w(\tau)$. Near the time of the arrival of the Rayleigh wave the infinity in $q(\tau)$, as well as in $w(\tau)$, varies with distance like $1/\sqrt{r}$, which is characteristic of surface waves.

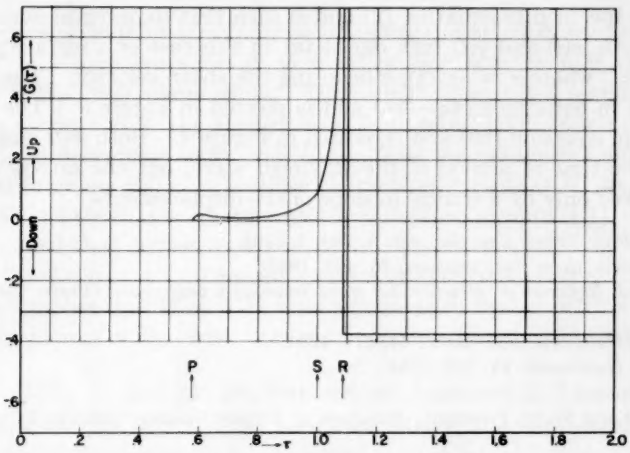


FIG. 3.—Vertical displacement at the surface $w(t)$ due to the application of a Heaviside unit pressure pulse $H(t)$ at the surface. $w(t) = -(Z/\pi\mu r)G(\tau)$, $\tau = (c_s t/r)$. Z (negative) is the surface integral of the applied pressure. $w(t)$ was computed from equations (45)–(48). P denotes arrival time of the compressional wave, S of the shear wave, and R of the Rayleigh wave.

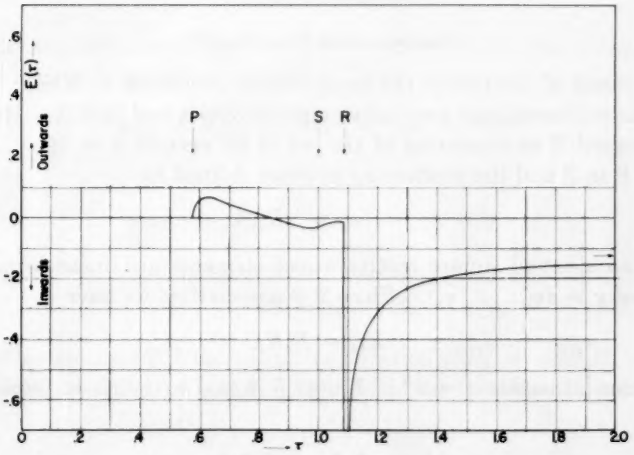


FIG. 4.—Horizontal displacement at the surface $q(t)$ due to the application of a Heaviside unit pressure pulse $H(t)$ at the surface. $q(t) = -(Z/\pi\mu r)E(\tau)$; $\tau = (c_s t/r)$ Z (negative) is the surface integral of the applied pressure. $q(t)$ was computed from equation (66). P, S, and R denote times of arrival of compressional, shear, and Rayleigh waves, respectively.

6. *Summary.*—Exact and closed expressions are derived for both the horizontal displacement $q(t)$ and the vertical displacement $w(t)$ of the surface of a uniform elastic half-space due to the application at the surface of a point pressure pulse varying with time like the Heaviside unit function $H(t)$. The applied pres-

sure pulse is specified in equation (2) and is such that its integral over the surface is finite. Both $w(t)$ and $q(t)$ turn out to be, in this case of a surface pulse, of the form $(1/r)f(\tau)$, where $\tau = (ct/r)$, c denoting the shear velocity. The solution for $w(t)$ is given in equations (45)–(48) and is plotted in Figure 3. The solution for $q(t)$ is given in equation (66) and is plotted in Figure 4. Both $w(t)$ and $q(t)$ become infinite at the time of arrival of the Rayleigh wave, but the arrival of the shear wave is marked only by a change in slope of the displacements.

[1] H. Lamb, *Phil. Trans. Roy. Soc.*, ser. A, **203,** 1, 1904.

[2] C. L. Perkeris, these PROCEEDINGS, **26,** 433, 1940.

[3] L. Cagniard, *Réflexion et réfraction des ondes seismiques progressives* (Paris: Gauthier-Villars, 1939).

[4] E. Pinney, *Bull. Seis. Soc. Amer.*, **44,** 571, 1954.

[5] C. H. Dix, *Geophysics*, **19,** 722, 1954.

[6] H. Bateman and C. L. Perkeris, *J. Opt. Soc. Amer.*, **35,** 655, 1945.

[7] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals* (Berlin: J. Springer, 1954), p. 225.

ON INVOLUTORIAL ALGEBRAS

BY A. A. ALBERT

UNIVERSITY OF CHICAGO*

Communicated May 18, 1955

The main result of this note is the proof of a conjecture of A. Weil.[1]

Let $\mathfrak{A}$ be an n-dimensional associative algebra over a real field $\mathfrak{F}$. It will be convenient to regard $\mathfrak{A}$ as consisting of the set of all vectors $x = (\xi_1, \ldots, \xi_n)$ with co-ordinates ξ_i in $\mathfrak{F}$ and the product xy as being defined by

$$xy = xR_y, \tag{1}$$

where R_y is an n-rowed square matrix whose elements are linear forms in the co-ordinates η_j of $y = (\eta_1, \ldots, \eta_n)$. Since $\mathfrak{A}$ is associative, we have

$$R_{xy} = R_x R_y. \tag{2}$$

A linear transformation $x \to x^*$ of $\mathfrak{A}$ over $\mathfrak{F}$ is said to define an *involution* over $\mathfrak{F}$ of $\mathfrak{A}$ if

$$(xy)^* = y^*x^*, \qquad (x^*)^* = x, \tag{3}$$

for every x and y of $\mathfrak{A}$. Assume that $x \to x^*$ is an involution of $\mathfrak{A}$ over $\mathfrak{F}$ and that there exists a linear functional $\delta(x)$ on $\mathfrak{A}$ to $\mathfrak{F}$ such that

$$\delta(xx^*) > 0 \tag{4}$$

for every nonzero x of $\mathfrak{A}$. Then we have the following result.

THEOREM 1. *The algebra $\mathfrak{A}$ is semisimple.*

For the radical $\mathfrak{N}$ of $\mathfrak{A}$ is the set of all properly nilpotent elements z of $\mathfrak{A}$. If z is in $\mathfrak{N}$, we have $(zx^*)^n = 0$ for every x of $\mathfrak{A}$, $[(zx^*)^n]^* = (xz^*)^n = 0$, and z^* is in $\mathfrak{N}$. Then $y = z + z^*$ is in $\mathfrak{N}$ for every z of N and $y^* = y$. If $y \neq 0$ for some

z in N, then $y^k = w \neq 0$ for some k, $w = w^*$, $w^2 = 0$, $ww^* = 0$, whereas $\delta(ww^*) > 0$. Hence $z^* = -z$ for every z of $\mathfrak{N}$, $w = zz^* = -z^2 = w^*$ is in $\mathfrak{N}$, $w = -w^* = w^*$, and $w = 0$, $\delta(w) > 0$ is impossible. Then $z = 0$, $\mathfrak{N} = 0$, $\mathfrak{A}$ is semisimple.

If a is any element of $\mathfrak{A}$, the algebra $\mathfrak{K}_r = \mathfrak{F}[a]$, of all polynomials in a with coefficients in $\mathfrak{F}$, is isomorphic to $\mathfrak{K}_0 = \mathfrak{F}[R_a]$. Then the minimum function of R_a is the minimum function $\phi(\xi)$ of a, and the characteristic roots of R_a are the roots of $\phi(\xi) = 0$. Evidently $\mathfrak{K}$ is isomorphic to the difference algebra $\mathfrak{F}[\xi] - (\phi)$, where (ϕ) is the principal ideal defined by $\phi = \phi(\xi)$.

THEOREM 2. *Let $a = a^*$. Then the complex characteristic roots of R_a are all real.*

For $a = a^*$ implies that $\mathfrak{B} = \mathfrak{F}[a]$ is an algebra such that $x \to x^*$ is the identity automorphism over $\mathfrak{F}$ of $\mathfrak{B}$. Also, $\delta(x)$ is a linear functional on $\mathfrak{B}$ to $\mathfrak{F}$ and

$$\delta(x^2) > 0 \tag{5}$$

for every nonzero x of $\mathfrak{B}$. By the argument above, the characteristic roots of R_a depend only on the minimum function of a, and this is the same when a is considered as an element of $\mathfrak{B}$ as when a is considered as an element of $\mathfrak{A}$. This reduces our study to the case where $\mathfrak{A} = \mathfrak{F}[a]$. By Theorem 1, this algebra has a unity quantity e.

Since $\delta(x)$ is linear, the function $\delta(xy)$ is a symmetric bilinear form on $\mathfrak{A} = \mathfrak{F}[a]$. Then

$$\delta(xy) = xSy', \tag{6}$$

where S is a symmetric matrix and y' is the transpose of $y = (\eta_1, \ldots, \eta_n)$. By relation (5), the matrix S is positive definite. Then

$$\delta(x) = \delta(xe) = \delta(ex) = xSe' = eSx'. \tag{7}$$

It follows that

$$\delta(xy) = (xy)Se' = xR_ySe' = xSy', \tag{8}$$

and so $Sy' = R_ySe'$,

$$yS = eSR_y'.$$

We now use the fact that $\mathfrak{A}$ is commutative to write $ySR_z' = eSR_y'R_z' = eS(R_zR_y)' = eSR_{zy}' = eSR_{yz}' = (yz)S = yR_zS$. This yields the fundamental relation

$$R_zS = SR_z' = (R_zS)'. \tag{10}$$

Since S is positive definite, we may write $S = PP'$, where P is a real nonsingular matrix. Then $R_zPP' = PP'R_z'$, $P^{-1}R_zP = P'R_z'(P')^{-1} = (P^{-1}R_zP)'$. We have proved that $P^{-1}R_aP$ is symmetric. Its characteristic roots are all real, so that the characteristic roots of the similar matrix R_a must all be real.

We shall say that a is *positive* if

$$a = a^* \neq 0, \qquad \delta(xax^*) \geq 0, \tag{11}$$

for every x of $\mathfrak{A}$. If $a = \sum\beta_i x_i \cdot x_i^* \neq 0$ for $\beta_i \geq 0$, then $xax^* = \sum\beta_i x x_i x_i^* x^* = \sum\beta_i y_i y_i^*$ and $\delta(xax^*) = \sum\beta_i\delta(y_iy_i^*) > 0$ if at least one $\beta_i \neq 0$. A. Weil conjectured that if $\mathfrak{F} = \mathfrak{R}$ is the field of all rational numbers, then a is positive if and

only if $a = \sum \beta_i \cdot x_i \cdot x_i^*$. We shall derive a somewhat stronger form of this result. We first prove

THEOREM 3. *Let a be positive. Then the nonzero characteristic roots of R_a are all positive.*

For we have already shown that we may restrict our attention to the case where $\mathfrak{A} = \mathfrak{F}[a]$. By hypothesis, $\delta(xax) \geq 0$ for every x of $\mathfrak{A}$, and so we see that $\delta(xax) = xaSx' = xR_aSx' \geq 0$ and the symmetric matrix R_aS of equation (10) is positive semidefinite. If x is a characteristic vector of R_a and α is a characteristic root such that $xR_a = \alpha x$, we have $\delta(xax) = \alpha xSx'$. But $xSx' > 0$, $xR_aSx' \geq 0$ for all real nonzero vectors x_1, and so $\alpha xSx' \geq 0$, $\alpha \geq 0$ as desired.

We now derive the final result.

THEOREM 4. *Let $\mathfrak{A}$ be an involutorial algebra over the field $\mathfrak{R}$ of all rational numbers, and let $\delta(x)$ be a functional satisfying relation* (4). *Then an element $a = a^*$ of $\mathfrak{A}$ is positive if and only if $a = x_1^2 + x_2^2 + x_3^2 + x_4^2$ for $x_i = x_i^*$ in $\mathfrak{R}[a]$.*

For $\mathfrak{R}[a]$ is semisimple and so may be written as the direct sum $\mathfrak{R}[a] = \mathfrak{R}[a_1] \oplus \ldots \oplus \mathfrak{R}[a_t]$ for algebraic fields $\mathfrak{R}[a_i]$ over R such that $a = a_1 + \ldots + a_t$. If $a_i = \sum_{j=1}^{4} x_{ij}^2$ then $a = \sum_{j=1}^{4} x_j^2$, where $x_j = \sum_{i=1}^{t} x_{ij}$. Also, $\delta(x_i a x_i) = \delta(x_i a_i x_i) \geq 0$ for every x_i of $\mathfrak{R}[a_i]$, $\delta(x_i^2) > 0$ for every $x_i \neq 0$ of $\mathfrak{R}[a_i]$. Thus we have reduced the proof of our theorem to the case where $\mathfrak{A} = \mathfrak{R}[a]$ is a totally real algebraic number field. Since a is totally positive, the quadratic form $f(y) = y_1^2 + y_2^2 + y_3^2 + y_4^2 - ay_5^2$ has the property that all algebraic number conjugates are indefinite. By a theorem of H. Hasse,[2] the form $f(y)$ is a null form, that is, $y_1^2 + y_2^2 + y_3^2 + y_4^2 = ay_5^2$ for y_i not all zero and in $\mathfrak{R}[a]$. But $y_5 \neq 0$, since otherwise $y_1^2 + \ldots + y_4^2 = 0$ for real y_i not all zero. Hence $a = x_1^2 + x_2^2 + x_3^2 + x_4^2$, with $x_i = y_i(y_5)^{-1}$.

* This paper was sponsored in part by the Office of Ordnance Research, United States Army, under Contract No. DA-11-022-ORD-1571.

[1] The conjecture is connected with Weil's attempt to generalize the results of H. Morikawa, "On Abelian Varieties," *Nagoya Math. J.*, **6**, 151–170, 1953. A special case of the theorem of this note is proved there, but the present proof is more general and of a more algebraic nature.

[2] *J. f. Math.*, **152**, 113–130, 1923.

FUNCTIONAL EQUATIONS IN THE THEORY OF DYNAMIC PROGRAMMING. II. NONLINEAR DIFFERENTIAL EQUATIONS

BY RICHARD BELLMAN

RAND CORPORATION, SANTA MONICA, CALIFORNIA

Communicated by Einar Hille, May 19, 1955

1. *Introduction.*—In the first paper of this series[1] (cf. also other papers[2]) we discussed the existence and uniqueness of solutions of the functional equation

$$f(P) = \max_{Q} f(T(P, Q)), \tag{1.1}$$

the basic functional equation of the theory of dynamic programming. In this paper we shall present the first of a series of investigations of various continuous analogues of the foregoing equation. Our results here concern the nonlinear system

$$\frac{dx_i}{dt} = \underset{q}{Max}\, f_i(x, t; q), \qquad x_i(0) = c_i, \qquad i = 1, 2, \ldots, n, \tag{1.2}$$

which we shall write in the vector form

$$\frac{dx}{dt} = \underset{q}{\text{Max}}\, f(x, t; q), \qquad x(0) = c. \tag{1.3}$$

The quantities x, f, and c are n-dimensional vectors, while q is an m-dimensional vector whose components may range over a finite or denumerable set, or continuum of values.

In subsequent papers we shall present analogous results for partial differential equations of parabolic type, for the first-order partial differential equations obtained from the calculus of variations,[3, 4] and for other classes of functional equations as well. A detailed exposition of the results of this note and a discussion of their relation to an interesting class of problems in the calculus of variations will appear elsewhere.

Equations of the above type appear in the study of continuous decision processes; some particular examples have been treated in unpublished notes by Copeland and Darling. In many ways these equations constitute natural generalizations of linear systems and thus play an important role in analysis, apart from their probabilistic applications. Furthermore, as we shall see below, some important types of equations, such as the Riccati equation, can be written in this form.

2. *A General Existence and Uniqueness Theorem.*—Let us first state a general result.

THEOREM 1. *Consider the system in equations* (1.2). *Let us assume that* $f(x, q, t)$ *satisfies the following conditions:*

a) $\|f(x, t; q) - f(y, t; q)\| \leq k(q, t)\|x - y\|$, *for* x, y, t *in the region* R *defined by the inequalities* $\|x - c\| \leq a$, $\|y - c\| \leq a$, $0 \leq t \leq t_0$, *for some nonzero constants* a *and* t_0.

b) The maximum value of $f_i(x, t; q)$ *is assumed for a function* $q_i = q_i(x, t)$ *which is uniformly bounded in* R.

c) The function $k(q, t)$ *is uniformly bounded for uniformly bounded* q *and for* t *in* R.

Then there is a unique solution of equation (1.3) *for* $0 \leq t \leq t_1$, *where* t_1 *is some nonzero constant less than* t_0. *This solution may be obtained as the limit of the sequence defined by*

$$\begin{aligned} x_0 &= c, \\ x_{n+1} &= c + \int_0^t \underset{q}{Max}\, f(x_n, s; q)\, ds. \end{aligned} \tag{2.2}$$

There are many interesting questions concerning the structure of the solution x and the policy function q which we do not have space to discuss here.

3. *Quasi-linear Systems.*—As usual, we can obtain a much more complete result for systems of linear type.

THEOREM 2. *Consider the system*

$$\frac{dx_i}{dt} = \max_q \left[b_i(t; q) + \sum_{j=1}^{N} a_{ij}(t; q)x_j \right], \qquad x_i(0) = c_i, \qquad i = 1, 2, \ldots, n, \quad (3.1)$$

where we assume that

$$\max_q |a_{ij}(t; q)|, \qquad \max_q |b_i(t, q)| \leq f(t), \quad (3.2)$$

with $f(t)$ integrable over any finite t-interval.

Then there is a unique solution for $t \geq 0$ which may be found by the same method of successive approximations described above.

In many cases we can determine the asymptotic behavior of the solution (see a forthcoming paper[5] for a discussion of this problem for difference equations). Results of similar type may be established for differential-difference equations and more general types of integro-differential equations.

4. *Approximation in Policy Space.*—The approximation technique used above was the classical one. Let us now describe another type. Given $q_0(t)$, for each index i, we may compute an initial approximation, x^0, using the equation

$$\frac{dx^0}{dt} = A(t; q_0)x^0 + b(q_0)x, \qquad x^0(0) = c. \quad (4.1)$$

To obtain a better approximation, we determine $q_1(t)$, again for each i, by the condition that $q_1(t)$ maximize the expression $\left[\sum_{j=1}^{N} a_{ij}(t; q)x_i^0 + b_i(t, q) \right]$. Having determined $q_1(t)$ in this way, we compute a second approximation, x^1, using equation (4.1), with q_0 replaced by q_1. We now continue in this fashion, computing a sequence of policy functions $\{q_n\}$ and a sequence of approximations $\{x^n\}$. This type of approximation we call "approximation in policy space".[4] Concerning this process, we can establish the following result:

THEOREM 3. *If the linear system*

$$\frac{dx}{dt} = A(t, q)x + f(t), \qquad x(0) = 0 \quad (4.2)$$

has, for any fixed q, the property that $f \geq 0$ for $t \geq 0$ implies that $x \geq 0$ for $t \geq 0$, then approximation in policy space yields monotone convergence to the solution.

In particular, there is always convergence when x is a scalar variable.

If $A(q)$ is a constant matrix for each allowable q, then a necessary and sufficient condition for the above condition to hold is $a_{ij} \geq 0$, for $i \neq j$; if $A(t, q)$ is variable, then a sufficient condition is $a_{ij}(t) \geq 0$, for $i \neq j$.[6]

5. *The Riccati Equation.*—An important tool in the qualitative and quantitative theory of the second-order linear differential equation $u'' + a(t)u' + b(t)u = 0$ is the transformation of the equation into the form

$$v' = -v^2 - a(t)v - b(t), \quad (5.1)$$

the Riccati equation, by means of the substitution $u = \exp(\int v\, dt)$. We now observe that the Riccati equation may be written in the form

$$v' = \operatorname*{Min}_{q} [q^2 + 2qv(t) - a(t)v - b(t)], \tag{5.2}$$

an equation of the above general type. This representation furnishes a new approach to the study of the solutions of equation (5.1). There are several other classes of equations of both analytic and physical interest to which this quasi-linearization may be applied. This will be discussed elsewhere.

[1] R. Bellman, "Some Functional Equations in the Theory of Dynamic Programming," these PROCEEDINGS, **39**, 1077–1082, 1953.

[2] R. Bellman, *An Introduction to the Theory of Dynamic Programming* (RAND Monograph R-245, 1953); "Some Functional Equations in the Theory of Dynamic Programming. I. Functions of Points and Point Transformations," *Trans. Am. Math. Soc.* (to appear).

[3] R. Bellman, "Dynamic Programming and a New Formalism in the Calculus of Variations," these PROCEEDINGS, **40**, 231–235, 1954.

[4] R. Bellman, "Monotone Convergence in Dynamic Programming and the Calculus of Variations," these PROCEEDINGS, **40**, 1073–1075, 1954.

[5] R. Bellman, "Quasi-linear Equations" (to appear).

[6] R. Bellman, I. Glicksberg and O. Gross, "Some Variational Problems Occurring in the Theory of Dynamic Programming," *Rend. Palermo* (to appear).

STATIONARY SPACE-TIME IN GENERAL RELATIVITY

BY S. BOCHNER

PRINCETON UNIVERSITY

Communicated May 20, 1955

In the notation of the recent book of A. Lichnerowicz[1] we take a V_4 with an everywhere regular hyperbolic local line element which, somewhat differently from the book, we denote by

$$\sum_{\alpha,\,\beta=0}^{3} \bar{g}_{\alpha\beta}\, dx^\alpha\, dx^\beta. \tag{1}$$

We assume that V_4 is stationary, as there defined, and this implies that V_4 is the topological product of V_3 with R: $(-\infty < t < \infty)$ and that on V_3 the line element (1) induces certain data which on V_3 are linked by the relation

$$\Delta\xi = -\xi R + \frac{1}{4}\xi^3 H^2, \tag{2}$$

with which we will be concerned exclusively. Underlying it is a positive-definite Riemannian metric,

$$\sum_{i,\,j=1}^{3} g_{ij}\, dx^i\, dx^j, \tag{3}$$

which is induced by expression (1). The quantity ξ is a scalar function which is positive throughout,

$$\xi > 0, \tag{4}$$

and $\Delta\xi$ is the Laplacean $g^{ij}\xi_{,i,\,j}$; R is another scalar, which is denoted in Lichnerowicz with indices because it is induced by the Ricci tensor $\bar{R}_{\alpha\beta}$ pertaining to expression (1) and which is such that, if

$$\bar{R}_{\alpha\beta} = 0$$

throughout, then also

$$R = 0 \tag{5}$$

thoughout; and, finally,

$$H^2 = H_{ij}H^{ij} \geq 0, \tag{6}$$

where H_{ij} is a certain skew tensor.

Definition 1: We will say that V_4 is *empty* if on V_3 we have

$$H = 0, \qquad \xi = \text{const.}, \tag{7}$$

and we will say that it is *semi-empty* if only

$$H = 0. \tag{8}$$

Remark 1: The space is empty if and only if expression (1) is locally Euclidean, but this will not concern us. Also, relation (2), always on a V_3, also occurs in some field theories with suitable interpretation of the symbols, whether the theory be based on a V_4 or on V_k, $k \geq 5$, and relation (7) or (8) always implies a certain manner of *emptiness*.

In what follows, we are assuming that V_3 is C^2, g_{ij} is C^1, ξ is C^2, R is C^0, and H_{ij} is C^0.

Important criteria for the occurrence of relation (7), or perhaps only (8), are due to, among others,[1] Levi-Civita, R. Serini, Ch. Racine, Lichnerowicz, Einstein and Pauli, and Yvonne Fourès and Lichnerowicz (for a recent result in a related but different direction see a paper by A. H. Taub[2]), and it is the purpose of the present paper to comment on one of these criteria by way of a generalization. The generalization as such has been given before[3] in our work on "Curvature and Betti numbers" but will be restated for the present context.

THEOREM 1. *Let V_3 be compact.* (*i*) *If we assume*

$$-\int_{V_3} \xi R \, dv \geq 0, \tag{9}$$

then V_4 is semi-empty, and equality holds:

$$-\int_{V_3} \xi R \, dv = 0. \tag{10}$$

(*ii*) *If, more precisely,*

$$-R \geq 0, \tag{11}$$

then V_4 is empty, and, in fact, equation (5) holds.

Proof: If we integrate equation (2), we obtain

$$\int_{V_3} \Delta\xi \, dv = -\int_{V_3} \xi R \, dv + \frac{1}{4}\int_{V_3} \xi^3 H^2 \, dv. \tag{12}$$

On the other hand, since V_3 is compact, we have[4]

$$\int_{V_3} \Delta\xi \, dv = 0, \tag{13}$$

and relations (6), (9), (12), and (13) imply (8) and (10) as claimed. If, more precisely, relation (11) holds, then (2) implies $\Delta\xi \geq 0$, and the desired conclusion $\xi = \text{const.}$ follows now either from the formula[4]

$$\int_{V_3} (\xi + C) \cdot \Delta\xi \, dv = -\int_{V_3} g^{ij} \xi_{,i} \, \xi_{,j} \, dv \tag{14}$$

or from the following local theorem due to E. Hopf.

LEMMA. *In a co-ordinate neighborhood V, if a function $\xi(P)$ of class C^2 satisfies the inequality $\Delta\xi \geq 0$ and if there is a fixed point P^0 in V such that $\xi(P) \leq \xi(P^0)$ everywhere in V, then we must have $\xi(P) = \xi(P^0)$ everywhere in V.*

We are now turning to comment on Theorem 1, in case V_3 is not (necessarily) compact.

Assumption 1: There is given on V_3 a group $\Gamma:(\gamma)$ of one-to-one continuous transformations (with fixed points, perhaps) having the following property. There exists a compact set S^0 in V_3 such that, corresponding to any point P^0 in V_3, there is an element γ^0 in Γ such that the transformed point $\gamma^0 P^0$ lies in S^0.

Definition 2: Let $f(P)$ be a continuous function on V_3, either a scalar or, more generally, any tensor density. We call it "almost periodic" (or almost automorphic), relative to the given group Γ, if every infinite sequence of elements $\{\gamma_n\}$ in Γ contains an infinite subsequence $\{\gamma_{np}\}$, $p = 1, 2, \ldots$, having the following property. Corresponding to any point, there is a co-ordinate neighborhood in which the sequence of functions $f(\gamma_{np}P)$ converges uniformly in each component, and, if we denote the limit function by $f^*(P)$, then, reversely, the sequence of functions $f^*(\gamma_{np}{}^{-1}P)$ converges, in the same fashion, *toward the function $f(P)$ from which we have started.*

Remark 2: The italicized part of definition 2 makes it unnecessary to require that (the space V_3 shall have uniform structure and that) the sequences shall converge uniformly in all of V_3.

It is easily seen that sums, products, and contractions are again almost periodic and that for an almost periodic scalar $\xi(P)$ the number

$$m = \sup_P \xi(P) \tag{15}$$

is finite (and also the inf.).

THEOREM 2. *Under assumption 1, if the functions*

$$g_{ij}, \; g^{ij}, \; \xi, \; \xi_{,i}, \; \xi_{,i,j}, \; R, \; H_{ij}, \tag{16}$$

are all almost periodic with respect to Γ, and if we have $R \leq 0$, then V_4 is empty.

Proof: Relation (11) implies $\Delta\xi \geq 0$. Now, if the supremum (15) is attained, that is, if there is a point P^0 for which $\xi(P^0) = m$, then the lemma implies $\xi = \text{const.}$ everywhere. But relations (2), (6), and (11) then also imply $H = 0$ as claimed.

If, however, it is not attained, then there exists a sequence of points $\{P_n\}$ for which $\xi(P_n) \to m$. By requirement 1 there exist elements $\gamma_n \in \Gamma$ such that $\gamma_n P_n \to S^0$. But S^0 is compact, and if we apply definition 2 judiciously, we may assume that the sequence $\{\gamma_n\}$ has the following properties simultaneously. We have $\gamma_n P_n \to P^0$, where P^0 is a point of S^0, and the 7 sequences (see expressions [16]),

$$g_{ij}(\gamma_n P_n),\ g^{ij}(\gamma_n P_n),\ \xi(\gamma_n P_n),\ \ldots,$$

all converge uniformly in the neighborhood of every point of V_3. If we denote the limit functions by asterisks, $g_{ij}{}^*$, g^{*ij}, ξ^*, R^*, etc., then we again have

$$\Delta^*\xi^* = -\xi^* R^* + \frac{1}{4}\,\xi^{*3} H^{*2},$$

and the function ξ^* does attain its supremum. Therefore, ξ^* is constant; but, by the reverse part of definition 1, the function ξ is likewise a constant, and this completes the proof of the theorem.

If V_3 is the Euclidean E_3 and Γ the group of all translations, then assumption 1 is fulfilled, with S being any one single point of E_3. Also, in this case a function $\xi(x)$, $x = (x^1, x^2, x^3)$, is almost periodic in the sense of definition 2 if it is almost periodic in the sense of Bohr,[5] that is, if and only if it is a limit, uniformly in E_3, of finite sums

$$\sum_{\lambda} a_{\lambda_1\lambda_2\lambda_3}\, e^{i(\lambda_1 x^1 + \lambda_2 x^2 + \lambda_3 x^3)}$$

for arbitrary real numbers λ_1, λ_2, λ_3. Theorem 2 now implies the following:

THEOREM 3. *Assume that V_3 is the Euclidean E_3 and that we are given one global line element* (3) *on it. If the individual functions*

$$g^{ij}(x),\ g_{ij}(x),\ \frac{\partial g_{ij}}{\partial x^k},\ \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l},\ \xi,\ \frac{\partial \xi}{\partial x^k},\ \frac{\partial^2 \xi}{\partial x^k \partial x^l},\ R,\ H_{ij},$$

are all almost periodic in the sense of Bohr, and if $R \leq 0$, then V_4 is empty space.

An almost periodic function $f(x)$ on a group has a mean $M\{f(x)\}$ which on E_3 can be evaluated as

$$M\{f(x)\} = \lim_{T\to\infty} \frac{1}{(2T)^3} \int_{-T}^{T}\int_{-T}^{T}\int_{-T}^{T} f(x^1, x^2, x^3)\, dx^1\, dx^2\, dx^3,$$

say. Now, for a domain D in E_3 with sufficiently smooth boundary B, Stokes's formula gives

$$\int_D \Delta\xi \cdot dv = \int_B \frac{\partial \xi}{\partial n}\, d\omega; \tag{17}$$

but if D is the cube $-T \leq x^i \leq T$, $i = 1, 2, 3$, then, for $T \to \infty$,

$$\int_B |d\omega| = 0(T^2), \tag{18}$$

and

$$\frac{d\xi}{dn} = 0(1), \tag{19}$$

and therefore

$$M\{\Delta\xi\} = 0. \tag{20}$$

However, relation (2) implies

$$M\{\Delta\xi\} = -M\{\xi R\} + \frac{1}{4} M\{\xi^3 H^2\}, \tag{21}$$

and

$$f \geq 0 \quad \text{and} \quad Mf = 0 \quad \text{imply} \quad f = 0. \tag{22}$$

Hence the following conclusion.

THEOREM 4. *If in Theorem* 3 *we replace relation* (11) *by the weaker assumption*

$$-M\{\xi R\} \geq 0, \tag{23}$$

then V_3 is semi-empty.

Remark 3: If in Theorems 3 or 4 the line element (3) is not global but only local, then we must assume that functions (16) are almost periodic in the sense of Bohr, and the conclusions remain in force. For Theorem 3 this is a direct consequence of Theorem 2, and for Theorem 4 this follows from the fact that, if we put

$$\Delta\xi \cdot dv = \Delta\xi \sqrt{g}\, dx^1\, dx^2\, dx^3,$$

then the integrand is almost periodic in the sense of Bohr, and similarly for the other integrands in equation (21).

A meaningful extension of Theorem 4 to a V_3 with a general group Γ is not easy to obtain; but we will make a statement, a somewhat technical one, for the case in which the topological quotient V_3/Γ is itself a (compact) manifold, which we will denote by U_3. In this case the group Γ is countable, and we enumerate its elements as a sequence $\gamma^0, \gamma^1, \gamma^2, \ldots$, in which γ^0 is the identity, and we decompose V_3 into a sequence of disjoint "fundamental domains" $U^0, U^1, U^2, \ldots$, which are such that γ^n transforms U^0 into U^n. They are not really "domains," that is, open sets, but their closures are (compact) closures of domains. From relation (2) we obtain

$$\int_{U^m} \Delta\xi\, dv = -\int_{U^m} \xi R\, dv + \frac{1}{4} \int_{U^m} \xi^3 H^2\, dv, \tag{24}$$

and we note that for any function $f(P)$ on V_3 we have

$$\int_{U^m} f(P)\, dv(P) = \int_{U^0} f(\gamma^m P)\, dv(\gamma^m P). \tag{25}$$

Now, the integrand on the right side is the product of $dx^1\, dx^2\, dx^3$ with $f(\gamma^m P)$ $(g(\gamma^m P))^{1/2}$, and, if the last function is almost periodic in our sense, then it is almost periodic on Γ in the sense of von Neumann, uniformly for P in U^0. Such a function has, again, an almost periodic mean $M\{\quad\}$, again with property (22), and equation (24) implies

$$M_\gamma\left\{\int_{U^0} \Delta\xi\, dv\right\} = -M_\gamma\left\{\int_{U^0} \xi R\, dv\right\} + \frac{1}{4} M_\gamma\left\{\int_{U^0} \xi^3 H^2\, dv\right\}. \tag{26}$$

Hence the following conclusion.

THEOREM 5. *If it is known that, in analogy to equation* (13), *we have*

$$M_\gamma\{\textstyle\int_{U^0} \Delta\xi\, dv\} = 0, \tag{27}$$

then the assumption

$$M_\gamma\{\textstyle\int_{U^0} \xi R\, dv\} \leq 0$$

implies that V_4 is semi-empty.

This theorem is useless unless equation (27) can be established, and, if one wishes to prove this by imitating the procedure used for Theorem 4, then the following features have to be verified, subject to assumptions. It is possible to arrange the ordering $\{\gamma^m\}$ in such a manner that for a sequence of integers $T_n \to \infty$ we have

$$M\{f(P)\} = \lim_{n\to\infty} \frac{1}{T_n} \sum_{m=1}^{T_n} f(\gamma^m P) \tag{28}$$

uniformly for P in U^0. Also, the union of sets

$$\sum_{m=0}^{T_n} U^m$$

constitutes a piece of space whose actual boundary consists of several sufficiently smooth surfaces whose two-dimensional induced surface area is $o(T_n)$ as $n \to \infty$.

Remark 4: Finally, we note that, with the aid of the mean (27), we may even obtain the statement on emptiness itself without requiring almost-periodicity at all. Assume that for a certain mean (28) we may go over from equation (25) to equation (26); that relation (22) holds; and that we have not only equation (27) but also

$$M_\gamma\{\textstyle\int_{U^0} \xi\cdot\Delta\xi\, dv\} = -M_\gamma\{\textstyle\int_{U^0} g^{ij}\,\xi_i\xi_j\cdot dv\}$$

in analogy to equation (14); then $-R \geq 0$ implies that V_4 is empty.

[1] *Theories relativistes de la gravitation et de l'électromagnétisme* (Paris: Masson & Cie, 1955).

[2] "Empty Space-Times Admitting a Three Parameter Group of Motions," *Ann. Math.*, **53**, 472–490, 1951.

[3] "Vector Fields and Ricci Curvature," *Bull. Am. Math. Soc.*, **52**, 776–797, 1946, esp. pp. 782–83; K. Yano and S. Bochner, *Curvature and Betti Numbers* (Princeton, N.J.: Princeton University Press, 1953), pp. 182–185.

[4] "Remark on the Theorem of Green," *Duke J.*, **3**, 334–338, 1937.

[5] "Beiträge zur Theorie der fastperiodischen Funktionen. II. Funktionen mehnerer Variablen," *Math. Ann.*, **96**, 383–409, 1926.

THE COHOMOLOGY RING OF G/T

By R. Bott and H. Samelson

University of Michigan

Communicated by C. Chevalley, May 12, 1955

1. In this note we describe a method for constructing the cohomology ring $H^*(G/T)$ with integral coefficients, for an arbitrary compact Lie group G, with maximal torus T. This refines results due to A. Borel,[1] who found the cohomology ring of G/T mod p under the assumption that G has no p-torsion. We make strong use of the fact that the cohomology ring of G/T with rational coefficients is generated by $H^2(G/T)$. On the other hand, our procedure demonstrates anew that G/T (as well as G/C, where C is the centralizer of any torus of G) is torsion-free.[2]

Our procedure is as follows: We construct a manifold B, whose cohomology ring we can determine in terms of the diagram of G; in particular, there is no torsion; dim B = dim G/T; and we construct a map f of B into G/T.

PROPOSITION I. *The map f of B into G/T has degree one.*

COROLLARY. *G/T has no torsion (since B has no torsion).*

THEOREM I. *$H^*(G/T)$ is isomorphic to the smallest subring of $H^*(B)$ which is additively a direct summand and which contains $f^*H^2(G/T)$.*

This is a consequence of Proposition I and the fact that $H^*(G/T)$ is rationally generated by $H^2(G/T)$.

To determine $H^*(G/T)$, we have then to describe $H^*(B)$ and to identify $f^*(H^2(G/T))$; this will be done in Proposition II.

2. We recall briefly the concepts connected with the diagram of G; we may and shall assume G to be semisimple. L is the Lie algebra of G, T a maximal torus, H the Lie algebra of T; dim $T = l$, dim $G = n = l + 2m$. The diagram Σ consists of the elements of H whose images in G under the exponential map are singular, i.e., have centralizers of dim $> l$; it consists of m families of parallel equidistant planes, which partition H into compact convex cells. The dimension of the centralizer corresponding to $x \in H$ is $l +$ twice the number of planes of Σ going through x. H carries a natural Euclidean metric, given by the Killing form; we denote the inner product by a dot. The lattice Λ is obtained by reflecting the origin o in all planes of Σ. Let Σ' be the subset of Σ through o. We denote the planes in Σ' by $P_1, \ldots, P_m$ *in the order in which* they meet some directed straight-line segment λ in H, which meets them all and does not go through o. Each P_i determines an element $\theta_i \in \Lambda$, obtained by reflecting o in the next plane parallel to P_i and such that $\theta_i \cdot \lambda > 0$. Under the exponential map $H \to T$, the segment from o to θ_i maps into a simple closed curve (closed 1-parameter group and geodesic) in T, whose homology class in $H_1(T)$ we denote by y_i. Let $\{t_\alpha\}$ be a subset of $\{y_i\}$, which is a base for $H_1(T)$ (corresponding to a fundamental system of roots). This defines integers $a_{i\alpha}$, $i = 1, \ldots, m$, $\alpha = 1, \ldots, l$, by

$$y_i = \Sigma a_{i\alpha} t_\alpha.$$

Finally, we recall the fact that the numbers $\alpha_{ij} = (2\theta_i \cdot \theta_j)/(\theta_j \cdot \theta_j)$ are integers, the Cartan integers of G.

3. We come now to the construction of B and f. Let C_i, $i = 1, \ldots, m$, be the component of the identity of the centralizer in G of the image, under the exponential map, of P_i. We form the products $W^k = C_1 \times \ldots \times C_k$, $k = 1, \ldots, m$. Since $T \subset C_i$ for all i, we can let $T^k = T \times \ldots \times T$ (k factors) operate on W^k on the right by the law

$$(c_1, \ldots, c_k) \cdot (t_1, \ldots, t_k) = (c_1 t_1, t_1^{-1} c_2 t_2, t_2^{-1} c_3 t_3, \ldots, t_{k-1}^{-1} c_k t_k).$$

(This skew operation was suggested to us from Morse theory considerations; it represents roughly variations of segments between successive singular points on a geodesic in G.) This defines W^k as a principal T^k-bundle; let B^k be the corresponding base space. One verifies easily that the projection $W^{k+1} \to W^k$ (by suppressing the last factor) induces a map of $B^{k+1} \to B^k$, which is a fibermap with a 2-sphere as fiber and which has a cross-section. This makes it possible, following Gysin,[3] to compute $H^*(B^{k+1})$ in terms of $H^*(B^k)$ and the characteristic class of an associated 1-sphere bundle; it is this class which brings the homology classes y_i and the integers α_{ij} into our argument. The space B promised above is B^m. Let $\tilde{f}$: $W^m \to G$ and f': $T^m \to T$ be defined by $\tilde{f}(c_1, \ldots, c_m) = c_1 c_2 \ldots c_m$, $f'(t_1, \ldots, t_m) = t_m$.

If we think of G as a principal T-bundle, with base G/T, then $\tilde{f}$ defines a bundle map of W^m into G over f' and induces a map f of $B^m = B$ into G/T; this is the map f of Proposition I.

PROPOSITION II. *With a suitable base* $\{x_1, \ldots, x_m\}$ *of* $H^2(B)$, *the algebra* $H^*(B)$ *becomes* $Z[x_1, \ldots, x_m]/I$, *where the numerator is the polynomial ring in the variables* $x_1, \ldots, x_m$ *over the integers* Z *and where* I *is the ideal generated by the elements*

$$\rho_k = x_k^2 + \sum_1^{k-1} \alpha_{ik} x_i x_k, \qquad k = 1, \ldots, m.$$

Further, $f^*(H^2(G/T))$ *is generated by the elements*

$$z_\alpha = \sum_1^m a_{i\alpha} x_i, \qquad \alpha = 1, \ldots, l.$$

4. The main step in the proof of Proposition I is the observation that the point T of G/T has exactly one inverse image point under f or that the inverse image of T under $\tilde{f}$ is precisely the set $T^m \subset W^m$. To prove this, we argue as follows: Let p_i be the point of intersection of λ with P_i; let p_0, resp. p^*, be the initial, resp. terminal, point of λ. We map W^m into the space Γ of polygons in L, starting at p_0, by the following law: $(c_1, \ldots, c_m)$ goes into the polygon $\{p_0p_1, Ad\ c_1(p_1p_2), Ad\ c_1c_2(p_2p_3), \ldots, Ad\ c_1c_2 \ldots c_m(p_mp^*)\}$. One verifies that this ordered collection of line segments in L is a polygon from p_0 to $Ad\ c_1c_2 \ldots c_m(p^*)$. Now the orbit of p^* under $Ad\ G$ can be identified with G/T, and with this identification the map $(c_1, \ldots, c_m) \to Ad\ c_1c_2 \ldots c_m(p^*)$ is the composition of $\tilde{f}$ with the projection $G \to G/T$. Since $Ad\ G$ leaves the Killing form invariant, all these polygons have the same length as λ. But the polygon corresponding to $(c_1, \ldots, c_m)$ has a corner at p_i unless $c_i \in T$; therefore, the inverse image of p^* is $T^m \subset W^m$, as was to be shown.

5. As an example of our method, consider the case of G_2. Here, for a suitable λ, $H^*(B^6)$ is described by the relations

$$\begin{aligned} x_1^2 &= 0, \\ x_2^2 &= x_1x_2, \\ x_3^2 &= (x_1 + 3x_2)x_3, \\ x_4^2 &= -(x_2 + x_3)x_4, \\ x_5^2 &= (x_1 - x_3 - 3x_4)x_5, \\ x_6^2 &= (x_1 + x_2 - x_4 - x_5)x_6, \end{aligned}$$

while $f^*H^2(G/T)$ is generated by $\{\alpha, \beta\}$, with

$$\begin{aligned} \alpha &= x_1 + 3x_2 + 2x_3 + 3x_4 + x_5, \\ \beta &= x_1 - 2x_2 - x_3 - x_4 + x_6. \end{aligned}$$

An admittedly tedious computation yields first of all that

$$\alpha^2 + 3\beta^2 + 3\alpha\beta = 0,$$

so that α^2 is divisible by 3 and $H^*(G_2/T)$ is rationally spanned by products of the form $\alpha\beta^k$ and β^k. Further relations are $\alpha^6 = \beta^6 = 0$; moreover, $\alpha\beta^5 = 2x_1x_2 \ldots x_6$ ($=$ twice the fundamental cocycle of B), and β^3 is the first element divisible by 2. A basis for $H^*(G_2/T)$ is then given by the elements

$$\{1, \alpha, \beta, \alpha\beta, \beta^2, \alpha\beta^2, \beta^3/2, \alpha\beta^3/2, \beta^4/2, \alpha\beta^4/2, \beta^5/2, \alpha\beta^5/2\}.$$

A routine computation with the spectral sequence of the fibering $(G_2, T, G_2/T)$ confirms the result of Borel[4] that $H^*(G_2)$ has a cyclic group of order two as torsion group in dimensions 6 and 9.

6. Let K be a connected subgroup of maximal rank in G. The inclusions $T \subset K \subset G$ induce a fibering f_k: $G/T \to G/K$ of G/T over G/K with fiber K/T. If $P_{\alpha_1}, \ldots, P_{\alpha_k}$ are the planes through o in Σ' which belong to K, set $X_k = x_{\alpha_1} x_{\alpha_2} \ldots x_{\alpha_k}$ in $H^*(B)$. Our imbedding f^* of $H^*(G/T) \to H^*(B)$ now defines a homomorphism χ_k: $H^*(G/T) \to Z$ by $\chi_k(u) =$ coefficient of X_k in the expansion of $f^*(u)$ in terms of the multilinear base in the x_i's.

By Poincaré duality, χ_k can be realized by a definite element $k \in H^*(G/T)$ once G/T is oriented and $H^{2m}(G/T)$ thus identified with Z.

PROPOSITION III. *f_k^* maps a fundamental cocycle of G/K onto k.*

COROLLARY. *If $k \in H^*(B)$ is not divisible by a prime p, then G/K has no p-torsion.*

This proposition follows from the fact that the image under f_k^* of the fundamental class of G/K is the Poincaré dual (in G/T) of the fiber K/T.

[1] A. Borel, "Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts," *Ann. Math.*, **57**, 115–207, 1953.

[2] R. Bott, "On Torsion in Lie Groups," these PROCEEDINGS, **40**, 586–588, 1954; A. Borel, "Kählerian Coset Spaces of Semisimple Lie Groups," these PROCEEDINGS, **40**, 1147–1151, 1954.

[3] W. Gysin, "Zur Homologie-Theorie der Faserungen und Abbildungen von Mannigfaltigkeiten," *Comm. Math. Helv.*, **14**, 61–122, 1941.

[4] A. Borel, "Sur l'homologie et la cohomologie des groupes de Lie compacts connexes," *Am. J. Math.*, **76**, 273–342, 1954.

GEOMETRIC ASPECTS OF CURRENTS AND DISTRIBUTIONS

BY JAMES EELLS, JR.

INSTITUTE FOR ADVANCED STUDY

Communicated by H. Whitney, April 28, 1955

The theory of currents[1] on a manifold M is a calculus of abstract linear functionals, based on a generalized Stokes's theorem. These functionals are defined on a certain topological linear space of exterior differential forms on M; analytical operations on the currents are defined by duality (i.e., by their effect on the differential forms).

In the present note we outline a procedure which permits a direct definition and geometric interpretation of the currents and of operations on currents. In the case of the currents of degree zero (i.e., the distributions) the definitions can be modified to give, for example, a theory of distributions on locally compact groups.

The essential parts of the material summarized here are contained in the author's doctoral dissertation, written under the direction of Professor Hassler Whitney. The present work rests heavily on the ideas developed by Whitney in his forthcoming book.[2]

1. Let M be an ∞-smooth (i.e., class C^∞) n-dimensional Riemannian manifold; actually, our results are independent of the Riemannian structure. Let $C_r(M)$ denote the linear space (over the real number field) of all ∞-smooth singular r-chains

of M. Given a tangent vector v at $p \in M$, we let $|v|$ denote its length; if $v(p)$ is an ∞-smooth vector field with compact support on M, we let $|v| = \sup\{|v(p)| : p \in M\}$. If σ is a singular r-simplex and v is a vector field (both ∞-smooth) on M, we define $T_v\sigma$ as the *translate of* σ *by* v (each point p of σ is moved the distance $|v(p)|$ along the trajectory determined by $v(p)$). $T_v\sigma$ is always defined if $|v|$ is small enough; we can extend the definition of T_v to an endomorphism of $C_r(M)$.

For each $m = 0, 1, 2, \ldots$ we define the seminorm λ_m on $C_r(M)$ as the largest seminorm λ satisfying $a, b, c_0, \ldots, c_m$:

a) $\lambda(\sigma) \leqslant |\sigma|_r$ for all r-simplexes σ on M, where $|\sigma|_r$ denotes the Lebesgue r-dimensional measure of σ on M;

b) $\lambda(\partial\tau) \leqslant |\tau|_{r+1}$ for any $(r+1)$-simplex τ on M, where $\partial\tau$ denotes the boundary r-chain of τ;

$$c_k)\ \lambda(\Pi_{i=0}^{k}(T_{v_i} - T_0)\sigma) \leqslant \frac{|v_0| \ldots |v_k| \, |\sigma|_r}{(r+1)\ldots(r+k+1)}$$

for all r-simplexes σ and all vector fields v_i, where $(T_{v_0} - T_0)(T_{v_1} - T_0)\sigma = T_{v_0+v_1}\sigma - T_{v_0}\sigma - T_{v_1}\sigma + \sigma$, etc.

Using the linear functionals on $C_r(M)$ which are bounded in λ_m, we can establish the following formula:

Given $A \in C_r(M)$, we consider all expressions of A of the form

$$A = \sum a_i\sigma_i + \sum b_i\partial\tau_i + \sum a_{0i}(T_{v_{0i}} - T_0)\sigma_{0i} + \ldots + \sum a_{mi}\,\Pi_{k=0}^{m}(T_{vki} - T_0)\sigma_{mi}.$$

Then, minimizing over all such expressions for A, we have

$$\lambda_m(A) = \inf\Big\{\sum|a_i| \, |\sigma_i|_r + \sum|b_i| \, |\tau_i|_{r+1} + \ldots + \sum \frac{|a_{0i}| \, |v_{0i}| \, |\sigma_{0i}|_r}{r+1} + \ldots + \sum \frac{|a_{mi}| \, |v_{0i}| \ldots |v_{mi}| \, |\sigma_{mi}|_r}{(r+1)\ldots(r+m+1)}\Big\}. \quad (1)$$

The case $m = 0$ was discovered by Whitney.[3]

We can use the theorem of Section 2 below to show that λ_m is a norm on $C_r(M)$; let $\mathfrak{C}_r^m(M)$ denote the completion of $C_r(M)$ in λ_m. From the definition of λ_m we observe that $\lambda_0(A) \geqslant \lambda_1(A) \geqslant \ldots$ for all $A \in C_r(M)$; we can identify $\mathfrak{C}_r^m(M)$ as a linear subspace of $\mathfrak{C}_r^{m+1}(M)$ for $m = 0, 1, 2, \ldots$. Let $\mathfrak{C}_r(M) = \cup_{m=0}^{\infty}\mathfrak{C}_r^m(M)$.

PROPOSITION. *For any* $A \in C_r(M)$ *we have* $\lambda_{m+1}(\partial A) \leqslant \lambda_m(A)$. *Hence we can extend* ∂ *to be a bounded mapping of* $\mathfrak{C}_r^m(M)$ *into* $\mathfrak{C}_{r-1}^{m+1}(M)$.

This can be proved with the help of formula (3) below.

2. Let $\mathfrak{C}^{m,r}(M)$ denote the conjugate space of $\mathfrak{C}_r^m(M)$; let $\mathfrak{C}^r(M) = \cap_{m=0}^{\infty}\mathfrak{C}^{m,r}(M)$. In the following theorem the term "derivative" refers to the Lie derivative $\nabla_v\xi$ of an ∞-smooth r-form ξ with respect to the vector field v.

THEOREM. *Given* $X \in \mathfrak{C}^r(M)$, *there is a unique* ∞*-smooth* r*-form* ξ *on* M *such that all derivatives of all orders* $\leqslant m$ *are uniformly bounded (by a constant depending on* m*), and for any* r*-simplex* σ *on* M *we have*

$$X \cdot \sigma = \int_\sigma \xi. \quad (2)$$

Conversely, every such r*-form on* M *defines, by formula* (2)*, an element of* $\mathfrak{C}^r(M)$.

The construction of ξ from a given X is fundamental in Whitney's theory see Whitney,[2, 3] where ξ is shown to exist under very weak hypotheses). In order to establish its differentiability and boundedness properties, we first show that if $X \in \mathfrak{E}^{m,r}(M)$, then the "$m$th Lipschitz constant"

$$\mathfrak{L}_m(X) = \sup \left\{ \frac{|X \cdot \Pi_{i=0}^{m}(T_{v_i} - T_0)\sigma|}{|v_0| \ldots |v_m| \, |\sigma|_r} \qquad (\text{all } \sigma \text{ and } v_i \neq 0) \right\}$$

is finite. It follows that ξ has bounded Lipschitz constants of orders $\leqslant m$, which fact in turn implies that ξ has derivatives of orders $\leqslant m$, each satisfying a Lipschitz condition of order 0 on M.

From this theorem it follows that if $X \in \mathfrak{E}^r(M)$, then its co-boundary δX (defined by $\delta X \cdot A = X \cdot \partial A$ for all $A \in C_r(M)$) is in $\mathfrak{E}^{r+1}(M)$ and corresponds to the exterior differential $d\xi$ of ξ. Also, if $X \in \mathfrak{E}^{m,r}(M)$, then

$$\mathfrak{L}_{m-1}(\delta X) \leqslant (r+1)\mathfrak{L}_m(X). \tag{3}$$

3. In case $r = 0$ and M is an open subset of Euclidean n-space, we can identify $\mathfrak{E}_0(M)$ with the space of *summable distributions*[4] of L. Schwartz. This can be seen from the following representation theorem or by studying the duality of the corresponding topological linear spaces. In general, $\mathfrak{E}_r(M)$ coincides with the space of *summable r-currents*. (Actually, de Rham treats the r-currents on M as $(n-r)$-forms with distributions as coefficients; we prefer the contravariant formulation.) The exterior differential of a current corresponds (except for sign) to the boundary of the associated chain.

THEOREM. *Given $A \in \mathfrak{E}_r(M)$, there exists a finite set of summable r-vector fields $\alpha_1, \ldots, \alpha_m$ and vector fields v_{ij} on M such that*

$$X \cdot A = \int_M \sum_i \nabla_{v_{i_1}} \ldots \nabla_{v_{ik}} \xi(p) \cdot \alpha_i(p) \, dp \tag{4}$$

for all $X \in \mathfrak{E}^r(M)$, using the correspondence (2); *the representation* (4) *is not unique. Conversely, every such representation defines an element of $\mathfrak{E}_r(M)$.*

4. The cup and cap products of algebraic topology can be defined in the spaces $\mathfrak{E}^r(M)$, $\mathfrak{E}_s(M)$ and correspond through formulas (2) and (4) to the exterior and interior products of the r-forms and s-vector fields on M. In particular, if $r = s = 0$, then the cap product corresponds to the "multiplication of distributions."[4]

5. If $M \times N$ denotes the Cartesian product of two manifolds M and N, then $\Sigma_{p+q=r}\mathfrak{E}_p(M) \otimes \mathfrak{E}_q(N)$ is contained in $\mathfrak{E}_r(M \times N)$; in fact, the tensor product $\otimes$ is a bounded operation in each λ_m.

In particular, if M is a Lie group, we can use the tensor product to define the *convolution product* $A * B \in \mathfrak{E}_{r+s}(M)$ of $A \in \mathfrak{E}_r(M)$ and $B \in \mathfrak{E}_s(M)$. If A and B are singular chains on M, then

$$X \cdot (A * B) = \int_A \int_B \xi(xy) \qquad \text{for all } X \in \mathfrak{E}^{r+s}(M). \tag{5}$$

The operation $*$ is bounded in λ_m and is associative. By taking for B an ∞-smooth function with compact support, we find that $A * B$ is an ∞-smooth summable r-vector field on M, for any $A \in \mathfrak{E}_r(M)$.

If $A = \alpha$ and $B = \beta$ are summable r- and s-vector fields on M, then $\alpha * \beta$ corresponds to the exterior product $\alpha \vee \beta$ as follows:

$$X \cdot (\alpha * \beta) = \int_{M \times M} \xi(xy) \cdot [\alpha(x) \vee \beta(y)]\, dx\, dy \qquad \text{for all } X \in \mathfrak{C}^{r+s}(M). \quad (6)$$

Formula (6) and Section 4 relate the convolution product (which coincides with the Pontrjagin product[5] on the homology classes) in the homology ring of a Lie group to the cup product in the cohomology ring.

6. With suitable modifications of the definition of λ_m, we can define families of seminorms on $C_r(M)$ which will give other spaces of currents on M.

7. In the results just outlined, an essential role is played by orientation properties if $r > 0$. In case $r = 0$, the theory takes on a simpler form and is applicable to other classes of spaces. For instance, if G is a locally compact group satisfying the first axiom of countability, then we can define an invariant metric on G; formulas for λ_m can be given as in Section 1 (condition b is not needed), replacing vector fields by group elements and setting $|p|_0 = 1$ for all $p \in G$. We can use the known structure of G to show that λ_m is a norm. In this way we can develop a theory of distributions on G similar to, but more general than, that of J. Riss.[6] The theory does not require the explicit notion of differentiation.

[1] Due to de Rham (see G. de Rham, *Variétés différentiables* [Paris: Hermann & Cie, 1955]).
[2] H. Whitney, *Geometric Integration Theory* (in preparation).
[3] H. Whitney, *Proc. Intern. Congr. Math.*, **1**, 245–256, 1950.
[4] L. Schwartz, *Théorie des distributions*, Vols. I and II (Paris: Hermann & Cie, 1950–1951).
[5] H. Hopf, *Ann. Math.*, **42**, 22–52, 1941.
[6] J. Riss, *Acta Math.*, **89**, 45–105, 1953.

GENERAL FACTORS OF AUTOMORPHY

By R. C. Gunning

PRINCETON UNIVERSITY

Communicated by S. Bochner

Let D be a simply connected analytic manifold, and let Γ be a group of analytic homeomorphisms of D onto itself, transformations with fixed points being permitted, such that (1) the orbit space D/Γ is compact; (2) each point $z \in D$ has a co-ordinate neighborhood U such that $TU \cap U \neq \phi$ for any $T \in \Gamma$ implies $TU = U$; (3) for each point $z \in D$ the set Γz has no limit points in D; for each pair of points $z_1, z_2 \in D$ either $\Gamma z_1 = \Gamma z_2$, or there exist neighborhoods U_1, U_2 of z_1, z_2 such that $\Gamma U_1 \cap \Gamma U_2 = \phi$; (4) there exists a Kaehler metric on D which is invariant under the group Γ.

Definition: A set of *factors of automorphy* for the group Γ on D, as defined by Bochner,[1] is a collection of functions $\{\nu_T(z)\}$ which are holomorphic on D, are indexed by the elements $T \in \Gamma$, and satisfy $\nu_{ST}(z) = \nu_S(Tz)\nu_T(z)$. We say that two sets of factors of automorphy $\{\mu_T(z)\}$ and $\{\nu_T(z)\}$ are *equivalent* if there exist complex constants c_T and a holomorphic nonvanishing function $h(z)$ on D such that $\mu_T(z) = c_T \nu_T(z)\, h(Tz)/h(z)$.

Consider any presentation $\Gamma \cong F/R$, where F is a free group on a set of generators $\{t_1, \ldots, t_q\}$. For each t_j select an arbitrary branch of the logarithms to define the single-valued holomorphic function $\sigma_{t_j}(z) = \log \nu_{T_j}(z)$, t_j corresponding to

the transformation T_j of Γ. Extend these functions to a set $\{\sigma_t(z)\}$ indexed by all $t \in F$ and satisfying $\sigma_{st}(z) = \sigma_s(Tz) + \sigma_t(z)$. Then, for any $t \in R$, $\hat{\sigma}(t) = 1/2\pi i\, \sigma_t(z)$ is an integer constant; this defines a homomorphism $\hat{\sigma}: R \to Z$, where Z is the additive group of the integers. The restriction of this homomorphism to the subgroup $R \cap [F, F]$ of R is called the *character class* of the factors $\{\nu_T(z)\}$; it is independent of the choice of the functions $\sigma_{T_j}(z)$.

THEOREM 1. *Two sets of factors of automorphy are equivalent if and only if they have the same character class.*

THEOREM 2. *If Γ contains no transformations with fixed points and Cousin's second problem is solvable on D, then, whenever a meromorphic function $f(z)$ on D satisfies $f(Tz) = \nu_T(z)f(z)$, the divisor of $f(z)$ considered as a two-cocycle of D/Γ belongs to the cohomology class determined by the character class of the factors $\{\nu_T(z)\}$ under the Hopf isomorphism[2] $R \cap [F, F]/[R, F] \cong H_2(D/\Gamma; Z)$.* (*Remark:* This theorem can be adjusted to include a result for groups admitting transformations with fixed points.)

Applying Theorem 1 when D is the entire complex affine space and Γ a group of translations, the following theorem of Appell[3] results:

THEOREM 3. *In the complex multiperiodic case all factors of automorphy are of the form $\nu_T(z) = exp\,(\sum_\nu a_T^\nu z_\nu + b_T)h(Tz)/h(z)$, where a_T^ν and b_T are complex constants and $h(z)$ is a nonvanishing entire function.*

On the other hand, when D is as in Poincaré's work, the following theorem results:

THEOREM 4. *If D is a bounded subdomain of the complex affine space and all closed Γ-invariant differential forms of type (1, 1) on D with integer periods on the two-cycles of Theorem 2 are multiples of the Ricci curvature form (which is always true in one complex variable, for instance), then all factors of automorphy are of the form $\nu_T(z) = c_T\,(J_T(z))^{m/M}\, h(Tz)/h(z)$, where c_T are complex constants, $J_T(z)$ are the complex Jacobian determinants of the transformations T, and $h(z)$ is holomorphic and nonvanishing in D.*

These fractional powers of the Jacobians in one complex variable have been studied by H. Petersson,[4] and this theorem shows that in a sense they are the only factors of automorphy that need to be envisaged in many contexts.

As a generalization of the previous theory, the concept of a set of *matricial factors of automorphy* can also be introduced; these are matrices $\{M_T(z)\}$ of holomorphic functions indexed by elements $T \in \Gamma$ and satisfying $M_{ST}(z) = M_S(Tz)M_T(z)$. Two such sets of factors $\{M_T(z)\}$ and $\{N_T(z)\}$ are called *equivalent* if there is a nonsingular holomorphic matrix function $H(z)$ on D such that $H(Tz) = M_T(z)\, H(z)\, N_T^{-1}(z)$. The simplest results are obtained in one complex variable. A set of matricial factors $\{M_T(z)\}$ is called *irreducible* if there exists no nonsingular holomorphic matrix function $F(z)$ such that the factors $F(Tz)M_T(z)F^{-1}(z)$ are of the form

$$\begin{pmatrix} A_T(z) & B_T(z) \\ 0 & C_T(z) \end{pmatrix}.$$

THEOREM 5. *In one variable all irreducible matricial factors of automorphy are equivalent to factors of the form $C_T m_T(z)$, where C_T are constant matrices and $m_T(z)$ are scalar factors of automorphy.*

[1] S. Bochner, "On Compact Complex Manifolds," *J. Indian Math. Soc.*, **11**, 1–21, 1947; "Functions in One Complex Variable as Viewed from the Theory of Functions in Several Variables," in *Lectures on Functions of a Complex Variable* (Ann. Arbor: University of Michigan Press, 1955).

[2] H. Hopf, "Fundamentalgruppe und zweite Bettische Gruppe," *Comm. Math. Helvi.*, **14**, 257–309, 1941.

[3] P. Appell, "Sur les fonctions périodiques de deux variables," *J. math. pures et appl.*, **7**, 157–219 1891.

[4] H. Petersson, "Zur analytischen Theorie der Grenzkreisgruppen," *Math. Ann.*, **115**, 23–67, etc. 1937.

---

*DIGITAL COMPUTATIONAL METHODS IN SYMBOLIC LOGIC, WITH EXAMPLES IN BIOCHEMISTRY**

By Robert S. Ledley

OPERATIONS RESEARCH OFFICE, JOHNS HOPKINS UNIVERSITY

Communicated by George Gamow, April 9, 1955

INTRODUCTION AND THE DIGITALIZATION

The formal theory of symbolic logic, as developed by Boole,[1] Peirce,[2] Jevons,[3] and Schröder[4] before 1900 and by Whitehead, Russell, Hilbert, Gödel, and others after the turn of the century, is universally recognized for its fundamental importance as a cornerstone of scientific method and thought. Yet it has remained largely a philosophic and esoteric realm of study, and there have existed no methods of actual straightforward computation for direct large-scale applications to specific types of realistic problems. The purpose of this paper is not to present any new results in the formal theory of symbolic logic, but rather to offer a new system of digitalized computational methods in the propositional calculus for application to the great number of practical nonnumerical problems that so frequently occur in science, industry, and government. The object was to formulate a logical "arithmetic" of extreme simplicity, which would provide, for this realm of nonnumerical problems, systematic methods of solution as simple, straightforward, and versatile as those of numerical analysis are for problems of a numerical nature. The theory of Boolean equations[5] becomes a special instance of the more general methods presented in this paper.

It is perhaps not generally realized how wide a range of problems can be attacked and solved by methods of symbolic logic and the related Boolean algebra, which is the calculus of sets and classes. Direct application of logic can always be an aid to deductive reasoning—such as determining consequences of given premises, rules, or axioms and making hypotheses or theorems from which the given premises or factual relationships can be deduced. Besides the use of logical propositional methods in problems concerned with sentences, such as analysis of military information reports and legal and insurance documents, there appear to be even more important applications to fields of operations research, biology, medicine, design of experiments, etc., where the utility of symbolic logic per se is not as immediately evident. Now that the results of the propositional calculus of symbolic logic are

made computationally feasible by these new digitalized techniques, it is hoped that they can be directed toward successful solution of many such problems—two examples of problems in biochemistry being given at the end of this paper. The digitalization of the methods in terms of the binary number system provides easy mechanization on existing high-speed electronic computers or on a special electronic digital logic machine.

This paper will contain no proofs, although the foundations of the methods are briefly indicated. Also, it is assumed that the reader is familiar with the methods of symbolic logic[6] and Boolean algebra[7] as given in the elementary reference texts.

REVIEW OF DEFINITIONS AND NOTATION

Propositions are sentences that can be called either *true* or *false* and are symbolized by A_r or X_r. Propositions can be combined by the *propositional operations* "$\cdot$", "$+$", and "$-$", where "$A_r \cdot A_s$" (read "A_r *and* A_s") is true when both A_r and A_s are true; "$A_r + A_s$" (read A_r *or* A_s) is the inclusive *or;* and "$\bar{A}_r$" is the negation of A_r (read "*not* A_r"). (The analogous set or class interpretation of the symbols as *intersection*, *union*, and *complementation*, respectively, is easily recognized.) An uncombined proposition is called an *elementary element*, while a combination of propositions by the propositional operations is called a *combined element* or *Boolean function*, this latter sometimes being denoted by $f_r(A_1, A_2, \ldots)$.

It is important to distinguish between two different meanings of truth. Some combined elements are true due to the form of the combination, independent of the truth values of the component parts, as, for example, $X_r + \bar{X}_r$ and $(\overline{X_r \cdot X_s}) + (X_r + \bar{X}_s) \cdot (\bar{X}_r + X_s)$. Such elements are called tautologically true, or simply *tautologies*, and are denoted by I. (The intrinsically false element is denoted by O. Hence $\bar{I} = O$.) On the other hand, a certain combined element, not a tautology, may be called true under the circumstances of a problem, this being called *factual* (sometimes *empirical*) truth. (We will not distinguish between the language and the metalanguage but will leave the distinction to be ascertained from the context.)

Two combined elements that are emphasized in all the original as well as in the modern works in logic are $X_r \cdot X_s + \bar{X}_r \cdot \bar{X}_s$ and $\bar{X}_r + X_s$. The former is read "X_r *is equivalent to* X_s" (in truth value), abbreviated by $X_r = X_s$; and when $X_r = X_s$, then $f(X_r, A_1, A_2, \ldots) = f(X_s, A_1, A_2, \ldots)$ for any f. The latter is read "X_r *implies* X_s" or "*if* X_r *then* X_s", abbreviated by $X_r \rightarrow X_s$. Note that this combination is true if X_r and X_s are both false, which might seem contrary to usual usage in language; but the proposition "If you can do that trick, then I'll eat my hat" actually has this meaning. In general, the difficulties in formulating a problem in symbolic logic are no more formidable than the precisely analogous task in mathematics and depend primarily on how well defined the problem is to begin with. However, only the computational aspects of already formulated problems are considered in this paper.

THE BASIS AND THE DIGITALIZATION

To every proposition A_r will be associated a binary number $\#A_r$ called the *designation number* of A_r; in particular, this holds for the elementary elements. The set of designation numbers for the elementary elements is called a *basis*, and one such basis for a system of three elementary elements A_1, A_2, A_3 is:

$$\begin{aligned} & {\scriptstyle 87654321} \\ \#A_1 &= 01010101 \\ \#A_2 &= 00110011 \\ \#A_3 &= 00001111 \end{aligned}$$

where the small numbers above the columns number the positions of the bits. In general, it can be shown that in a system of n elementary elements the designation numbers will have 2^n positions. A basis is distinguished by the fact that the columns represent the 2^n possible combinations of 0, 1 taken n at a time. Thus there are $2^n!$ bases; the basis shown will be used throughout this paper due merely to its visual simplicity of pattern. The relation of a basis to the familiar truth tables is evident. It will be convenient to denote the bit in the ith column and rth row of a basis by $b[A_1, \ldots, A_s]^i_r$. For example, in our basis $b[A_1, A_2, A_3]^5_2 = 1$.

The propositional operations are interpreted in terms of the designation numbers in a manner reminiscent of vector addition as follows, where $\#A^i_r$ represents the bit 0 or 1 in the ith position of $\#A_r$:

$\#(A_r + A_s)^i = \#A^i_r + \#A^i_s$ according to the rule $1 + 1 = 1 + 0 = 0 + 1 = 1$, $0 + 0 = 0$;

$\#(A_r \cdot A_s)^i = \#A^i_r \cdot \#A^i_s$ according to the rule $1 \cdot 0 = 0 \cdot 1 = 0 \cdot 0 = 0$, $1 \cdot 1 = 1$;

and $\#\bar{A}^i_r = 0, 1$ when $\#A^i_r = 1, 0$, respectively.

The operation "+" is called logical addition, "·" logical multiplication, and "–" inversion. Thus the designation number of a combined element with respect to a particular basis is obtained by performing the operations indicated by the combination. For example, with respect to the basis given above, $\#(\bar{A}_1 + A_2 \cdot A_3) = 10101010 + (00110011) \cdot (00001111) = 10101010 + 00000011 = 10101011$. The set of all the 2^{2^n} possible designation numbers forms a representation of a free Boolean algebra. (Note that for elementary elements, $\#A^i_r = b[A_1, \ldots, A_s]^i_r$, of course.)

For the reverse process—namely, given the designation number, to find its propositional representation—note that every product of all the elementary elements or their negations, called an *elementary product*, has a single unit, i.e., $\#(A_1 \cdot A_2 \cdot A_3) = 00000001$, $\#(\bar{A}_1 \cdot A_2 \cdot A_3) = 00000010$, etc., for each of the 2^3 such products. Hence a propositional representation called the *disjunctive normal form* is obtained by forming the sum of those elementary products corresponding to the units of the designation number. For example, $01100010 = \#(A_1 \cdot \bar{A}_2 \cdot \bar{A}_3 + \bar{A}_1 \cdot A_2 \cdot \bar{A}_3 + \bar{A}_1 \cdot A_2 \cdot A_3)$. Other propositional representations can be systematically generated, such as the simplest-sum-of-products form, simplest-product-of-sums form, and the conjunctive normal form.[8]

Important observations are that $\#I$ consists only of units, i.e., $\#I^i = 1$ for all i. Hence $\#O^i = 0$ for all i. Also, $X^i_s = X^i_r$ if and only if $\#X_s = \#X_r$ for all i. In addition, $X_r \rightarrow X_s$ if and only if $\#X_s$ has units in at least the same positions as does $\#X_r$; i.e., if $\#X^i_r = 0$, then $\#X^i_s = 0, 1$, but if $\#X^i_r = 1$, then $\#X^i_s = 1$.

COMPUTATIONAL METHODS

CONSTRAINTS

The concept of constraints is fundamental for the computational methods. Constraints are logical relations between the elementary elements which are to be

considered as tautologically true, due to intrinsic or stated circumstances of a problem. In other words, constraints are factually true combined elements which are to remain true throughout a problem and therefore, with respect to this problem, can be considered as tautologies. A systematic computational method will be given that automatically guarantees the maintenance of any desired constraints throughout the solution of a problem. After constraints have been applied, a *constrained propositional calculus* results, which can be interpreted in terms of a nonfree Boolean algebra.

In particular, if H is an automorphism of a Boolean algebra B, such that "+", "$\cdot$", and "$-$" are preserved [i.e., for $X_r, X_s, \epsilon B$, $H(X_r + X_s) = H(X_r) + H(X_s)$, $H(X_r \cdot X_s) = H(X_r) \cdot H(X_s)$, and $H(\bar{A}_r) = \overline{H(A_r)}$], and if $F(A_1, A_2, \ldots) \epsilon B$ represents a constraint, then we desire that $H(F) = I$. Under this automorphism, equivalence classes are formed by the set $\{F\}$ together with the sets $\{X\}_i$, where $(F + X_r) \epsilon \{F\}$ for all $X_r \epsilon B$, this being the kernel of H, and for $X_s \epsilon \{F\}$, $X_s \epsilon \{X\}_i$ if $H(X_s \cdot F) = X_i$. The equivalence classes form the elements of the nonfree Boolean *quotient algebra* B/F, when the operations between the classes are defined by $\{X\}_r + \{X\}_s = \{X_r + X_s\}$, $\{X\}_r \cdot \{X\}_s = \{X_r \cdot X_s\}$, and $\overline{\{X_r\}} = \{\bar{X}_r\}$. If $\#F$ has u units, then there will be $2^{(2^n - u)}$ distinct elements in each equivalence class and 2^u equivalence classes all together. This quotient algebra automatically satisfies the constraints and represents a constrained propositional calculus.

Given the constraints $F_1, F_2, \ldots, F_r$, the computational method consists in first forming $\prod_{\mu=1}^{\mu=r} (\#F_\mu)$ and then reducing the reference basis to include only those columns j for which $\prod_{\mu=1}^{\mu=r} (\#F_\mu^j) = 1$ (i.e., those columns corresponding to the unit positions of the product of the constraints). All computations with respect to this reduced basis will automatically have $\#F_\mu = \#I$ ($\mu = 1, 2, \ldots, r$), as desired. For example, the constrained basis $b_c[A_1, A_2, A_3]$ that will make A_1, A_2, A_3 mutually exclusive and exhaustive as is often desired in problems concerning classification of subject matter, namely, $A_1 \cdot A_2 = O$, $A_1 \cdot A_3 = O$, $A_2 \cdot A_3 = O$, $A_1 + A_2 + A_3 = I$, becomes

$$\begin{matrix} & & & & 764 \\ \#A_1 & = & & & 100 \\ \#A_2 & = & & & 010 \\ \#A_3 & = & & & 001 \end{matrix}$$

For this constraint, A_1, A_2, A_3 are called the components of the three-component proposition A and form an example of an n-component proposition.

ANTECEDENCE AND CONSEQUENCE SOLUTIONS

The majority of logical problems involve given or accepted premises, hypotheses, rules, or other logical relationships which are the *given equations* and essentially comprise the statement of the problem. There are two types of solutions to a set of given equations, the *antecedence solutions* and the *consequence solutions*. Antecedence solutions are hypotheses or theories from which the given equations can be deduced; consequence solutions can be deduced from the given equations. In other words, the truth of the antecedence solutions is sufficient for the truth of the given equations, but the truth of the consequence solutions is necessary for the

truth of the given equations. If the given equations are true, then the consequent solutions are true, while the antecedence solutions may or may not be true; but the truth of the given equations can be deduced from the hypotheses embodied in the antecedence solutions, this latter being the method for theory construction.

However, to produce just one antecedence or one consequence solution to given equations is usually trivial; hence logical problems always require solutions of a specified form or solutions involving only certain specified elementary elements or both. Occasionally solutions with the required properties do not exist, and the problem is extended to determine under what conditions such solutions do exist for the given equations. (It is of historical interest to note that the problem of solution to equations as posed by Schröder and others is merely a special case of the general theory presented here, being antecedence solutions of the particular form $X_s = f_s$.)

More specifically, a complete set of antecedence solutions for given equations is any set of constraints F that makes each of the equations true in the constrained system B/F (i.e., that maps each equation into I). All combined elements that become true (are mapped into I) in the constrained system B/E, the constraints being the given set of equations E, are consequence solutions. With these definitions in mind, the problem is summarized in Figure 1. The given equations E_1, ..., E_r are combined elements or functions of the elementary elements A_1, ..., A_i, X_1, ..., X_k. The specified forms F_1, ..., F_s for the antecedence or consequent

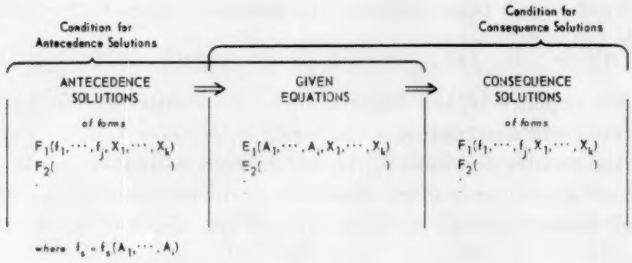


FIG. 1

solutions are, in general, known functions of X_1, ..., X_k and of f_1, ..., f_j, where the f_j are as yet unknown functions of A_1, ..., A_i. The object is to determine explicitly the functions f_j or, if they do not exist, conditions for their existence. Summarizing, then, computational methods for determining the existence of antecedence or consequence solutions of specified forms and, if they exist, for generating all such solutions, or, if not, for determining conditions of existence, will be presented.

The fundamental formulas for determining antecedence and consequence solutions are the following:

ANTECEDENCE

$$\boxed{(F_{jk}) \otimes (E_{kl}) = (\bar{R}_{jl})}$$

CONSEQUENCE

$$\boxed{(\bar{F}_{jk}) \otimes (E_{kl}) = (\bar{R}_{jl})}$$

where (F_{jk}), (E_{kl}). (R_{jl}) are Boolean matrices (of elements 0, 1 only), and $\otimes$ represents logical matrix multiplication [i.e., if $(P_{jk}) \otimes (Q_{kl}) = (S_{jl})$, then $\sum_k P_{jk}Q_{kl} = S_{jl}$ with logical multiplication and addition]. The "$-$" represents inversion of

each individual element of the matrix. The formation of (E_{ki}) and (F_{jk}), and the interpretation of (R_{ji}) as the desired solutions, proceed as follows.

The Formation of Matrices (E_{ki}) *and* (F_{jk}).—Two bases enter into this calculation namely, $b[A_1, \ldots, A_i, X_1, \ldots, X_k]$ and $b[f_1, \ldots, f_j, X_1, \ldots, X_k]$, where the f_j's are considered as elementary elements and both bases have the usual pattern, the last rows being formed by the X_k's. Then form $\prod_{\mu=1}^{\mu=r} (\#E_\mu)$ with respect to $b[A_1, \ldots, A_i, X_1, \ldots, X_k]$ and $\prod_{\nu=1}^{\nu=s} (\#F_\nu)$ with respect to $b[f_1, \ldots, f_j, X_1, \ldots, X_k]$. Now separate the positions of the designation number $\prod_{\mu=1}^{\mu=r} (\#E_\mu)$ into 2^k successive groups of positions, with 2^i positions in each group. Index the groups from right to left by k ($k = 1, \ldots, 2^k$) and the positions in each group by i ($i = 1, \ldots, 2^i$). The rows of the matrix (E_{ki}) are simply these respective groups of positions. Similarly, $\prod_{\nu=1}^{\nu=s} (\#F_\nu)$ is separated into 2^k groups, this time of 2^j positions, the groups being indexed by k and the positions in each group by j ($j = 1, \ldots, 2^j$). The columns of the matrix (F_{jk}) are simply these respective groups of positions.

In more mathematical terms, the formula for an element E_{ki} of the matrix (E_{ki}) is $E_{ki} = \prod_{\mu=1}^{\mu=r} (\#E_\mu^p)$ for $(p-1) = (k-1)2^i + (i-1)$. Similarly, $(F_{jk}) = \prod_{\mu=1}^{\mu=r} (\#F_\mu^q)$ for $(q-1) = (k-1)\,2^j + (j-1)$. (Note, of course, that $\prod_{\mu=1}^{\mu=r} (\#E_\mu^p) = \#(\prod_{\mu=1}^{\mu=r} E_\mu)^p$ and similarly for $\prod_{\nu=1}^{\nu=s} (\#F_\nu^q)$).

The Interpretation of (R_{ji}).—Two bases are consulted in this process, namely, $b[f_1, \ldots, f_j]$ and $b[A_1, \ldots, A_i]$, both written in the usual pattern. The desired solution is computed by means of the result array, which consists of an as yet empty array with 2^i columns and j rows; the columns are indexed by i from right to left, the rows corresponding to $f_1, \ldots, f_j$ assigned from top to bottom. Now, having computed (R_{ji}) by the proper formula, consider those pairs of indices j, i, for which $R_{ji} = 1$: place the jth column of $b[f_1, \ldots, f_j]$ in the ith column of the result array. The rows of the result array thus filled are the designation numbers of the corresponding $f_1, \ldots, f_j$ with respect to the basis $b[A_1, \ldots, A_i]$ and hence give the desired explicit functions $f_1(A_1, \ldots, A_i), \ldots, f_j(A_1, \ldots, A_i)$.

In more mathematical terms, $\#f_m(A_1, \ldots, A_i)^i = b[f_1, \ldots, f_j]_m^j$ for those j for which $R_{ji} = 1$, and $m = 1, \ldots, j$, where $\#f_m(A_1, \ldots, A_i)$ is interpreted with respect to $b[A_1, \ldots, A_i]$.

Multiple Sets of Solutions.—The occurrence of several columns of $b[f_1, \ldots, f_j]$ in a single result-array column indicates multiple solutions, for each such column determines a different set of solutions. Hence the total number of different sets of solutions is the product of the number of columns of $b[f_1, \ldots, f_j]$ in each array column. If a column of the result array has no basis column in it, then no solutions exist at all.

Conditions for the Existence of Solutions.—Consider the row vector (V_j) comprising 2^j units. If no solution exists, compute the row vector (C_i) from the formula $(V_j) \otimes (R_{ji}) = (C_i)$. Then (C_i) is the designation number of the desired condition for the existence of solutions, as referred to $b[A_1, \ldots, A_i]$. In order to impose this condition new bases constrained by the condition must now replace $b[A_1, \ldots,$

$A_i, X_1, \ldots, X_k]$ and $b[A_1, \ldots, A_i]$, and the solution is computed exactly as above, except that for "2^i" now read "u", where u is the number of units in (C_i).

A Simple Example.—Consider the especially simple equation of historical interest discussed at length by Schröder, namely, $A_1X + A_2\bar{X} = O$, to be solved for antecedence solutions of the form $X = f(A_1, A_2)$. The computations proceed as follows:

$b[f, X]$:

$$\begin{array}{rcl} \#f & = & 0101 \\ \#X & = & 0011 \\ \hline \#(f = X) & = & 1001 \\ \begin{matrix} j \\ k \end{matrix} & & \begin{matrix} 2\,1\,2\,1 \\ 2\;\;1 \end{matrix} \end{array}$$

$b[A_1, A_2, X]$:

$$\begin{array}{rcl} \#A_1 & = & 01010101 \\ \#A_2 & = & 00110011 \\ \#X & = & 00001111 \\ \hline \#(A_1X + A_2\bar{X} = O) & = & 11001010 \\ \begin{matrix} i \\ k \end{matrix} & & \begin{matrix} 4\,3\,2\,1\,4\,3\,2\,1 \\ 2\;\;\;\;1 \end{matrix} \end{array}$$

whence $(F_{jk}) = \left(\begin{smallmatrix}10\\01\end{smallmatrix}\right)$ and $(E_{ki}) = \left(\begin{smallmatrix}1100\\1010\end{smallmatrix}\right)$. Substituting in the antecedence formulas, $(F_{jk}) \otimes (E_{ki}) = \left(\begin{smallmatrix}10\\01\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix}0011\\0101\end{smallmatrix}\right) = \left(\begin{smallmatrix}0011\\0101\end{smallmatrix}\right) = (\bar{R}_{ji})$, whence $(R_{ji}) = \left(\begin{smallmatrix}1100\\1010\end{smallmatrix}\right)$. To interpret (R_{ji}),

$b[f]$:

$$\begin{array}{rcl} \begin{matrix} j \\ \\ \end{matrix} & & \begin{matrix} 2\,1 \\ \\ \end{matrix} \\ \#f & = & 01 \end{array}$$

Result array:

i	4	3	2	1
$\#f$	0, 1	0	1	none

Thus no solution exists. The condition for the existence of a solution is computed by $(V_j) \otimes (R_{ji}) = (11) \otimes \left(\begin{smallmatrix}1100\\1010\end{smallmatrix}\right) = (1110)$ to be interpreted with respect to $b[A_1, A_2]$: $\#A_1 = 0101$, $\#A_2 = 0011$, whence $1110 = \#(\bar{A}_1 + \bar{A}_2)$, i.e., $\bar{A}_1 + \bar{A}_2 = I$ in order that $A_1X + A_2\bar{X} = O$ have a solution.

The constrained basis $b[A_1, A_2, X]$ becomes $\#A_1 = 010\ 010$, $\#A_2 = 001\ 001$, $\#X = 000\ 111$, whence $\#(A_1X + A_2\bar{X} = O) = 110101$ and $(E_{ki}) = \left(\begin{smallmatrix}110\\101\end{smallmatrix}\right)$. Then $\#(R_{ji}) = \left(\begin{smallmatrix}1\,1\,0\\1\,0\,1\end{smallmatrix}\right)$, and the result array is

i	3	2	1
$\#f$	0, 1	0	1

which, interpreted in terms of the constrained bases $b_c[A_1, A_2]$: $\#A_1 = 010$, $\#A_2 = 001$, is $\#f = 001 = \#A_2$ and $\#f = 101 = \#\bar{A}_1$. Hence, under the proper conditions, the given equation has two solutions, $X = A_2$ and $X = \bar{A}_1$.

Finding Antecedence and Consequence Solutions Involving Only Specified Elementary Elements.—Let $A_1, \ldots, A_i$ be the specified elementary elements, all the rest being included in the $X_1, \ldots, X_k$. Then all the computations proceed as above. For example, consider the problem of determining consequence solutions of $A_1X + A_2\bar{X} = O$ involving only A_1 and A_2. As above, $(E_{ji}) = \left(\begin{smallmatrix}1100\\1010\end{smallmatrix}\right)$, but the desired form is simply f, and so, since $\#f = 0101$ as referred to bases $b[f, X]$, $(F_{jk}) = \left(\begin{smallmatrix}00\\11\end{smallmatrix}\right)$. Thus $(\bar{F}_{jk}) \otimes (E_{ki}) = \left(\begin{smallmatrix}11\\00\end{smallmatrix}\right) \otimes \left(\begin{smallmatrix}1100\\1010\end{smallmatrix}\right) = \left(\begin{smallmatrix}1110\\0000\end{smallmatrix}\right) = (\bar{R}_{ji})$. Therefore, $(R_{ji}) = \left(\begin{smallmatrix}0001\\1111\end{smallmatrix}\right)$, whence

$$\begin{array}{rcl} \begin{matrix} j \\ \\ \end{matrix} & & \begin{matrix} 2\,1 \\ \\ \end{matrix} \\ \#f & = & 01 \end{array}$$

i	4	3	2	1
$\#f$	1	1	1	0, 1

or $f = \bar{A}_1 + \bar{A}_2$ and $f = I$. In other words, $\bar{A}_1 + \bar{A}_2 = I$ or, equivalently, $A_1 \cdot A_2 = O$ is the desired nontrivial solution.

Special Theorems.—Solutions of the form $f_m = X_m$, $m = 1, \ldots, k$ are particularly important and occur frequently. For antecedence solutions of this form, (F_{jk})

is the identity matrix, and hence $(E_{ki}) = (R_{ji})$ (as was seen in the above example), enabling (R_{ji}) to be written down directly. In addition, it can be shown that if, to a set of given equations, both antecedence solutions and consequence solutions of this form exist, then there is only one such set of solutions (i.e., no multiple solutions), and it is the same for both cases. Also, if only one consequence (antecedence) solution of this form exists, then it is also an antecedence (consequence) solution of this form.

LOGICAL DEPENDENCE (CONDITIONS) AND LOGICAL INDEPENDENCE

Conditions on the Solutions.—Often it is desired to find only those solutions $f_1, \ldots, f_j$ which satisfy given conditions or constraints $G(f_1, \ldots, f_j)$ or, in other words, only those $f_1, \ldots, f_j$ for which the given combined element or function $G(f_1, \ldots, f_j) = I$. The computational procedure is similar to calculating solutions with conditions on the $A_1, \ldots, A_i$, except that now bases constrained by $G(f_1, \ldots, f_j)$ replace $b[f_1, \ldots, f_j, X_1, \ldots, X_k]$, and $b[f_1, \ldots, f_j]$ (where $\#G(f_1, \ldots, f_j)$ is determined with respect to $b[f_1, \ldots, f_j]$); and for "2^j" now read "u", where u is the number of units in $\#G(f_1, \ldots, f_j)$. All sets of solutions thus calculated will automatically satisfy the given conditions.

Logically Independent Solutions.—Since $f_1 = f_1(A_1, \ldots, A_i), \ldots, f_j = f_j(A_1, \ldots, A_i)$, it can happen that these equations imply a logical relation between $f_1, \ldots, f_j$. For example, if $f_1 = A_1 \cdot A_2$ and $f_2 = A_1 + A_2$, then $f_1 \rightarrow f_2$. However, often a prime requirement of sets of solutions for a problem is that they be logically independent, that is, no relations of the form $G(f_1, \ldots, f_j) = I$ exist other than tautologies. Then $f_1, \ldots, f_j$ will be logically independent when the result array has at least 2^j different columns. Three cases arise: $j = i$, $j < i$, $j > i$. If $j = i$, then there can be no repeated columns in the result array for independent solutions. For $j < i$, 2^j different columns must appear in the array for independence. For $j > i$ there can be no independent set of solutions.

Determining the Logical Dependence between a Given Set of Solutions.—Given a set of solutions, it is often desired to find the logical dependence between them, if any exists at all. The computational method is to consider the result array for this given set of solutions $f_1, \ldots, f_j$. Rewrite the array, omitting any duplicate columns, and adjoin to this (on the right) those columns necessary to make the completed array a basis, say $b'[f_1, \ldots, f_j]$, not necessarily of the usual pattern. Then the designation number with zeros in the positions corresponding to the adjoined columns, units elsewhere, produces the desired conditions between $f_1, \ldots, f_j$ when referred to $b'[f_1, \ldots, f_j]$.

EXAMPLES IN BIOCHEMISTRY

ENZYME CHEMISTRY

In biochemistry an enzyme often cannot be isolated easily, and therefore the experiments performed often will involve several other enzymes. Consequently a combination of reactions must be observed. In such a complex situation the ordinary simple logical analysis usually used in experimental science is found inadequate. In these cases the logical computational methods presented above can be extremely useful for evaluating experimental results as well as for planning future experiments to yield the maximum information.

Suppose that a chemist were studying enzymes, A_1, A_2, A_3 in relation to reactions X_1, X_2, X_3. Suppose that he has done the following four experiments: (E_1)—A solution containing none of A_1, A_2, A_3 produced reaction X_2 but not X_1, and not X_3. (E_2)—The solution contained A_1 and either A_2 or A_3, or both; he could not be sure. The reaction was neither X_2 nor X_1 and X_3 together. (E_3)—The solution had A_2 but not A_1, or did not have A_2 but had A_3. Reactions X_1 and X_2 occurred, or reaction X_3 occurred but X_1 did not. (E_4)—The solution was obtained from a source that had A_3 and A_1 or A_2 or both, or else had neither A_1 nor A_3; the solution turns color, which means X_1 does not take place or both X_2 and X_3 do.

The antecedence problem is: What theories about the enzymes associated with each reaction will explain the experimental results? The consequence problem is: What combinations of enzymes are necessary for each reaction to take place?

The experiments and their designation numbers with respect to the usual basis are as follows:

E_1: $\bar{A}_1 \cdot \bar{A}_2 \cdot \bar{A}_3 = \bar{X}_1 \cdot X_2 \cdot \bar{X}_3$

0111 1111 0111 1111 1000 0000 0111 1111 0111 1111 0111 1111 0111 1111 0111 1111

E_2: $A_1 \cdot (A_2 + A_3) = \bar{X}_2 \cdot (\overline{X_1 \cdot X_3})$

0001 0101 0001 0101 1110 1010 1110 1010 0001 0101 1110 1010 1110 1010 1110 1010

E_3: $\bar{A}_1 \cdot A_2 + \bar{A}_2 \cdot A_3 = X_1 \cdot X_2 + \bar{X}_1 \cdot X_3$

1101 0001 1101 0001 1101 0001 0010 1110 0010 1110 1101 0001 0010 1110 0010 1110

E_4: $A_3 \cdot (A_1 + A_2) + \bar{A}_1 \cdot \bar{A}_3 = \bar{X}_1 + X_2 \cdot X_3$

1010 0111 0101 1000 1010 0111 0101 1000 1010 0111 0101 1000 1010 0111 1010 0111

Thus $\prod_{\mu=1}^{\mu=4} (\#E_\mu) =$ 0000 0001 0001 0000 1000 0000 0000 1000 0000 0100 0100 0000 0010 0010 0010 0010

For the antecedence problem the form of the solution is $X_1 = f_1$, $X_2 = f_2$, $X_3 = f_3$. Recall for antecedence solutions of this special form that $(E_{ki}) = (R_{ji})$.

Hence

$(E_{ki}) = (R_{ji}) =$

	8765	4321
8	0000	0001
7	0001	0000
6	1000	0000
5	0000	1000
4	0000	0100
3	0100	0000
2	0010	0010
1	0010	0010

from which the solutions are read:

$b[f_1, f_2, f_3]$:

j	87654321
$\#f_1$ =	01010101
$\#f_2$ =	00110011
$\#f_3$ =	00001111

Result array:

i	8	7	6	5	4	3	2	1
$\#f_1$	0	1	1, 0	1	1	0	1, 0	0
$\#f_2$	1	0	1, 1	0	1	0	1, 1	0
$\#f_3$	0	1	1, 1	0	0	1	1, 1	0

Hence there are four sets of antecedence solutions of the form $X_s = f_s$, namely,

$$\begin{array}{lll}
X_1 = f_1, \text{ where } f_1 = A_1\cdot\bar{A}_3 + \bar{A}_1\cdot A_3, & \bar{A}_3\cdot(A_1 + A_2) + \bar{A}_1\cdot\bar{A}_2\cdot A_3, & \\
X_2 = f_2, \text{ where } f_2 = \bar{A}_1 \quad , & \bar{A}_1 \quad , & \\
X_3 = f_3, \text{ where } f_3 = A_1\cdot\bar{A}_2 + \bar{A}_1\cdot A_2, & A_1\cdot\bar{A}_2 + \bar{A}_1\cdot A_2 \quad , & \\
 & \bar{A}_3\cdot(A_1 + A_2) + \bar{A}_1\cdot A_3, & A_1\cdot\bar{A}_3 + \bar{A}_1\cdot\bar{A}_2\cdot A_3; \\
 & \bar{A}_1 \quad , & \bar{A}_1; \\
 & A_1\cdot\bar{A}_2 + \bar{A}_1\cdot A_2 \quad , & A_1\cdot\bar{A}_2 + \bar{A}_1\cdot A_2.
\end{array}$$

The first two sets are logically independent. The last two sets of solutions imply that the reactions are not independent but that $X_2\cdot X_3 \rightarrow X_1$, and $X_1\cdot X_2\cdot X_3 = O$, respectively. For example, we write the designation numbers of the last solution, cross off the repeating column, adjoin the missing column and obtain:

$$\begin{array}{rcll}
\#(A_1\cdot\bar{A}_3 + \bar{A}_1\cdot\bar{A}_2\cdot A_3) & = & 0101 & 1001, \\
\#\bar{A}_1 & = & 1010 & 1010, \\
\#(A_1\cdot\bar{A}_2 + \bar{A}_1\cdot A_2) & = & 0110 & 0110, \\
 & & 1111 & 11\ 1\ 0 = \#(f_1\cdot f_2\cdot f_3 = O), \text{ or } X_1\cdot X_2\cdot X_3 = O.
\end{array}$$

For the consequence problem, the form of the solution and corresponding designation numbers with respect to $b[f_1, f_2, f_3, X_1, X_2, X_3]$ are as follows:

$$\begin{array}{ll}
\#(X_1 \rightarrow f_1) & 1111\ 1111\ 0101\ 0101\ 1111\ 1111\ 0101\ 0101\ 1111\ 1111\ 0101\ 0101\ 1111\ 1111\ 0101\ 0101 \\
\#(X_2 \rightarrow f_2) & 1111\ 1111\ 1111\ 1111\ 0011\ 0011\ 0011\ 0011\ 1111\ 1111\ 1111\ 1111\ 0011\ 0011\ 0011\ 0011 \\
\#(X_3 \rightarrow f_3) & 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 1111\ 0000\ 1111\ 0000\ 1111\ 0000\ 1111\ 0000\ 1111 \\
\prod_{\nu=1}^{\nu=4} (\#F_\nu) = & 1111\ 1111\ 0101\ 0101\ 0011\ 0011\ 0001\ 0001\ 0000\ 1111\ 0000\ 0101\ 0000\ 0011\ 0000\ 0001
\end{array}$$

hence

$$(F_{jk}) = \begin{pmatrix} 1000 & 0000 \\ 1100 & 0000 \\ 1010 & 0000 \\ 1111 & 0000 \\ 1000 & 1000 \\ 1100 & 1100 \\ 1010 & 1010 \\ 1111 & 1111 \end{pmatrix} \text{ and } (\bar{F}_{jk}) \otimes (E_{ki}) = (\bar{R}_{ji}), \text{ whence } (R_{ji}) = \begin{pmatrix} 0000 & 0001 \\ 0001 & 0001 \\ 1000 & 0001 \\ 1001 & 1001 \\ 0000 & 0101 \\ 0101 & 0101 \\ 1000 & 0101 \\ 1111 & 1111 \end{pmatrix},$$

whence the result array:

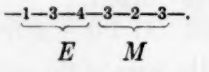
i	8	7	6	5	4	3	2	1
$\#f_1$	1010	11	1	1111	11	1010	1	ϕ
$\#f_2$	1111	10	1	1010	11	1100	1	ϕ
$\#f_3$	1100	11	1	1100	10	1111	1	ϕ

Since the most significant set of solutions for this form is the minimum set, i.e., the solutions with the most zeros, we choose

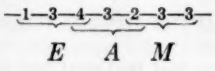
$$\begin{array}{lll}
\#f_1 = 0111\ 1010 = \#[\bar{A}_3\cdot(A_1 + A_2) + \bar{A}_1\cdot A_3], & & X_1 \rightarrow \bar{A}_3\cdot(A_1 + A_2) + \bar{A}_1\cdot A_3; \\
\#f_2 = 1010\ 1010 = \#\bar{A}_1, & \text{or} & X_2 \rightarrow \bar{A}_1; \\
\#f_3 = 0110\ 0110 = \#[A_1\cdot\bar{A}_2 + \bar{A}_1\cdot A_2], & & X_3 \rightarrow A_1\cdot\bar{A}_2 + \bar{A}_1\cdot A_2.
\end{array}$$

THE PROTEIN DECODING PROBLEM

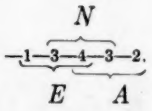
According to present views,[9] hereditary informations are stored in chromosomes in the form of long sequences of four different units (known as bases) constituting the molecules of nucleic acid. In the process of protein synthesis, these polynucleotide sequences must be translated into the corresponding polypeptide sequences of twenty different amino acids. Thus there must exist a mechanism which carries through such translations in a unique way, producing a long word written in a twenty-letter alphabet for each long number written in the four-digital system. Since twenty different triplets can be formed (with disregard to order) out of four different elements, it is inviting to associate each amino acid in the resulting polypeptide chain with a group of three bases in the original polynucleotide sequence. One can make different assumptions as to the type of correlation involved. For example, we can assume that while one amino acid is defined by three neighboring bases, its neighbor is given by the next three bases:



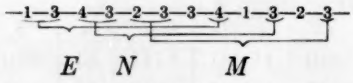
Or one can make the assumption of overlapping triplets, in which case each base is participating in the choice of more than one amino acid. For example, we may have the scheme



or the more restrictive scheme



The possibility is also not excluded that because of twisting of the chain, non-neighboring bases participate in the choice, as in the following:



In all overlapping translation codes, one should expect a stronger or weaker inter-symbol correlation, i.e., some restriction in the choice of neighboring amino acids. Thus, comparing these restrictions with actually observed sequences of amino acids in protein chains, one should be able to find the correct coding procedure and the one-to-one correspondence between the twenty amino acids and the twenty hypothetical triplets of bases. Since there are $20! = 2.3 \times 10^{17}$ such one-to-one correspondences (which equals the number of seconds in the age of the universe), the straightforward test of all possibilities is out of the question, and it is desirable to find a systematic and feasible procedure for solving the problem (with the aid of high-speed electronic computing machines).

Use of the Logical Computational Methods Developed Above.—To illustrate the method of solution, it is best to consider a much simpler case which makes more

transparent the computational methods involved and can easily be extended to the complex problem given above. Suppose that there were only three kinds of amino acids and only three types of base triplets allowed. Let the rules of combination involve groups of three consecutive triplets, and let the first position of the group be denoted by A, the second by B, and the third by C. In addition, the subscript 1, 2, or 3 will denote which one of the three allowed triplets is in the position, i.e., A_2 stands for "The first position has triplet 2". Hence A_s, B_s, and C_s, for $s = 1, 2, 3$, are each three component propositions. Similarly, let X_r, Y_r, Z_r, where $r = 1, 2, 3$, represent the positions of one of the three amino acids, considered in consecutive groups of three. Let the hypothetical conditions on the sequences of triplets be:

$$\begin{aligned} A_1 \cdot B_2 &\rightarrow C_3 + C_2 \\ A_1 \cdot B_3 &\rightarrow C_2 + C_3 \\ A_2 \cdot B_1 &\rightarrow C_1 + C_3 \\ A_2 \cdot B_3 &\rightarrow C_1 + C_3 \\ A_3 \cdot B_1 &\rightarrow C_1 + C_2 \\ A_3 \cdot B_2 &\rightarrow C_1 + C_2 \\ A_1 \cdot B_1 &\rightarrow C_1 + C_3 \end{aligned} \qquad \begin{aligned} A_1 \cdot B_2 &\rightarrow C_1 + C_2 \\ A_1 \cdot B_3 &\rightarrow C_1 + C_3 \\ A_2 \cdot B_1 &\rightarrow C_1 + C_2 \\ A_2 \cdot B_2 &\rightarrow C_1 + C_2 \\ A_2 \cdot B_3 &\rightarrow C_2 + C_3 \\ A_3 \cdot B_2 &\rightarrow C_2 + C_3 \\ A_3 \cdot B_3 &\rightarrow C_1 + C_3 \end{aligned}$$

Suppose that the experimental results gave the following conditions on the amino acids:

$$X_3 \cdot Y_3 \rightarrow Z_2 + Z_3, \qquad X_1 \cdot Y_1 \rightarrow Z_1 + Z_2, \qquad X_2 \cdot Y_2 \rightarrow Z_1 + Z_2.$$

The problem is to find antecedence solutions of the form $X_r = f_r$, $Y_r = g_r$, and $Z_r = h_r$ for $r = 1, 2, 3$. Introducing the following shorthand: 1 represents the column $\begin{smallmatrix}1\\0\\0\end{smallmatrix}$, 2 the column $\begin{smallmatrix}0\\1\\0\end{smallmatrix}$, and 3 the column $\begin{smallmatrix}0\\0\\1\end{smallmatrix}$, since A_s, B_s, C_s (and X_r, Y_r, Z_r) are three component propositions, find for $b[A_s, B_s, C_s]$ (or $b[f_r, g_r, h_r]$):

i (or j):	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
$\#A_s$ (or $\#f_r$) =	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3	1	2	3
$\#B_s$ (or $\#g_r$) =	1	1	1	2	2	2	3	3	3	1	1	1	2	2	2	3	3	3	1	1	1	2	2	2	3	3	3
$\#C_s$ (or $\#h_r$) =	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2	3	3	3	3	3	3	3	3	3

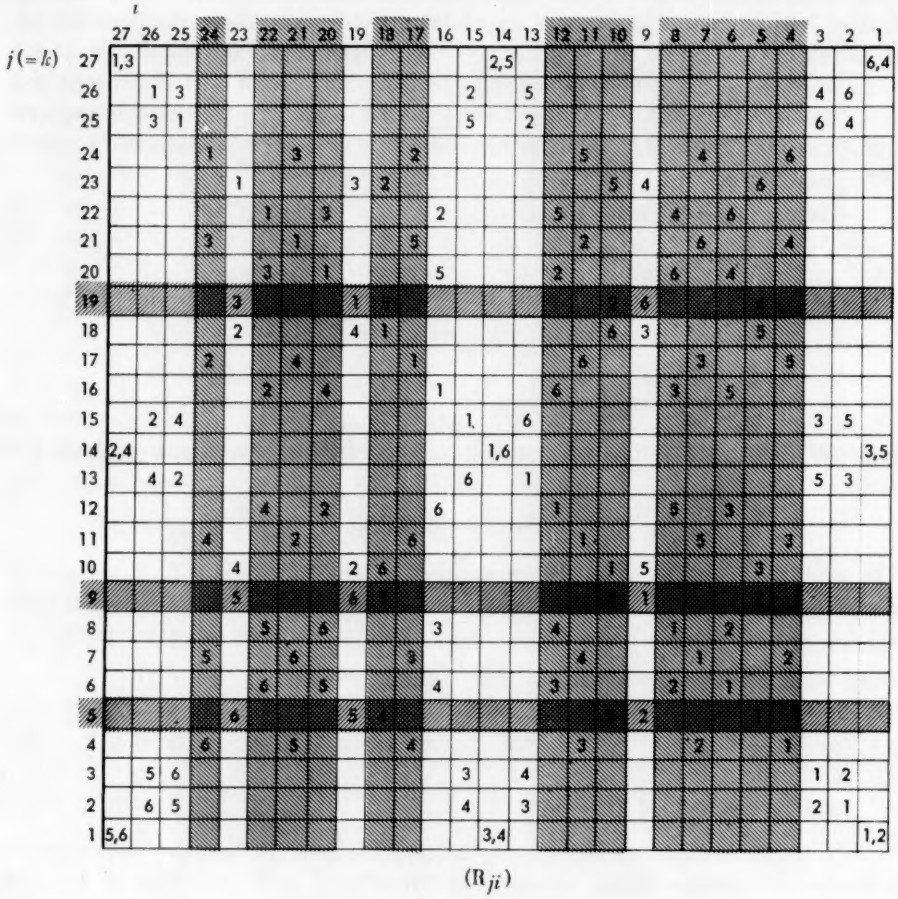
However, observe that we really know all possible answers, because there must be a one-to-one correspondence between the components of X_r and those of A_s; and, in addition, the same correspondence must hold between Y_r and B_s, and Z_r and C_s. Hence the 3! possible solutions are as follows:

Solution number:	1	2	3	4	5	6
$X_1 =$	A_1,	A_2,	A_1,	A_3,	A_2,	A_3
$X_2 =$	A_2,	A_1,	A_3,	A_1,	A_3,	A_2
$X_3 =$	A_3,	A_3,	A_2,	A_2,	A_1,	A_1

Letting $P = 121233$, $Q = 213312$, $R = 332121$, the result array for the solutions with respect to $b[A_s, B_s, C_s]$ can be written as follows:

i	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
$\#f_r$	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R	P	Q	R
$\#g_r$	P	P	P	Q	Q	Q	R	R	R	P	P	P	Q	Q	Q	R	R	R	P	P	P	Q	Q	Q	R	R	R
$\#h_r$	P	P	P	P	P	P	P	P	P	Q	Q	Q	Q	Q	Q	Q	Q	Q	R	R	R	R	R	R	R	R	R

where this is to be interpreted as $3! = 6$ solutions (*not* 6^{27} solutions according to the notation used above), where each solution corresponds to the same respective column for $i = 1, \ldots, 27$. Hence (R_{ji}) can be constructed, where we put the number of each solution in the proper R_{ji} element. Thus we obtain Figure 2.



(R_{ji})

FIG. 2

However, this matrix (R_{ji}) would be the result with no conditions on A_s, B_s, C_s, X_r, Y_r, Z_r. Adding the conditions to $b[A_s, B_s, C_s, X_r, Y_r, Z_r]$ amounts to eliminating some columns (corresponding to the zeros of the conditions on A_s, B_s, C_s) and some rows (corresponding to zeros of those on X_r, Y_r, Z_r) as indicated in Figure 2. Now only those solutions remain that are contributed to by *every* column. Hence, solution 1 is eliminated, for it does not appear in column 19 or column 9, and so forth. Only solution 4 is not eliminated and hence, under the given and experimental conditions, it is the only valid hypothesis.

Now we are able to sketch briefly the feasible method of solving this problem by means of computers.[10] First, there is a method for putting the $n!$ permutations of $1, \ldots, n$ into one-to-one correspondence with $1, \ldots, n!$, so that the correspond-

ence can be easily accomplished in either direction. Thus these integers will number the solutions uniquely. Second, there is also a systematic procedure for determining the solution number, given i and j $(= k)$. So choose some allowed column i_0, and find the solution numbers for all j that have not been eliminated due to constraints. Solution numbers not found by this step cannot be valid solutions, for they do not appear in this column i_0. Third, there is a procedure for determining j, given i and the solution number. So for all other allowed columns i, and for each solution number just obtained, try to determine j. Those solutions for which no allowed j can be found for every allowed i cannot be valid solutions. The remaining solutions are valid.

Of course, for $3! = 6$ solutions it is easiest to try them all. However, the feasible method just presented is intended for choosing one out of the 20! possible solutions of the original problem. The (R_{ji}) matrix in this case is $8{,}000 \times 8{,}000$, but, when a few hundred conditions are applied to A_s, B_s, C_s, X_r, Y_r, Z_r, it will reduce to about $5{,}000 \times 5{,}000$. It is estimated that the complete calculation should take a computer no more than a hundred hours. On the other hand, to try 20! solutions, a computer put to work in the days of the Roman Empire, at a rate of one million solutions per second, 24 hours a day, all year round, would not yet be close to finishing the job.

The author gratefully acknowledges the assistance and guidance of Dr. George Gamow, particularly with respect to the general description of the Protein Decoding Problem. The kind encouragement of Dr. Nicholas M. Smith and the Operations Research Office is especially appreciated.

* This paper is an excerpt of a technical memorandum which is being published by the Operations Research Office and is part of an investigation of methods for applying symbolic logic to operations research problems.

[1] George Boole, *George Boole's Collected Logical Works* [1859], ed. P. E. B. Jourdain (Chicago and London, 1916), Vols. 1 and 2.

[2] C. S. Peirce, "On the Algebra of Logic," *Am. J. Math.*, **3**, 15–57, 1880.

[3] W. S. Jevons, *Pure Logic* [1864] *and Other Minor Works*, ed. R. Adamson and H. A. Jevons (London and New York, 1890), pp. 1–77.

[4] E. Schröder, *Algebra der Logik* (Leipzig, 1890), Vol. 1.

[5] G. Birkhoff, "Lattice Theory," *Am. Math. Soc. Coll.* (New York, 1948).

[6] H. Reichenbach, *Elements of Symbolic Logic* (New York: Macmillan Co., 1947).

[7] G. Birkhoff and S. Mac Lane, *A Survey of Modern Algebra* (New York: Macmillan Co., 1948).

[8] Robert S. Ledley, "Mathematical Foundations and Computational Methods for a Digital Logic Machine," *J. Operations Research Soc. Amer.*, Vol. **2**, No. 3 (August, 1954); Robert S. Ledley, *Bull. Am. Math. Soc.*, **60** (No. 4), 378, 1954; **61** (No. 1), **50**, **52**, 1955; Robert S. Ledley, *New Computational Methods in Symbolic Logic for Operations Analysts: An Introductory Manual* (Operations Research Office Technical Memorandum), Vols. 1–3 (in publication).

[9] J. D. Watson and F. H. Crick, *Nature*, **171**, 964, 1953; G. Gamow, *Nature*, **173**, 318, 1954; *Kgl. Danske Videnskab Selskab Biol. Medd.*, Vol. **22**, No. 3, 1954; G. Gamow, A. Rich, and M. Yčas, chap. 00 in *Advances of Biological and Medical Physics* (New York: Academic Press, 1955) (in press); F. Lipmann, *Currents in Biochemistry* (in press).

[10] The details can be found in Vol. 2 of Ledley, *New Computational Methods.*

*THE BEHAVIOR OF A COMPLEX FUNCTION AT INFINITY**

By Karl Menger

ILLINOIS INSTITUTE OF TECHNOLOGY

Communicated by Lars V. Ahlfors, May 11, 1955

Traditionally, the behavior at ∞ of a complex function f is defined as the behavior at 0 of the function obtained by substituting the -1st power into f. This definition adequately describes the class of values $f(z)$ for large $|z|$. For instance, the range near ∞ of the identity function j (whose value for any z is $j(z) = z$) coincides with the range near 0 of the function j^{-1}. But that definition does not in any way describe the structural behavior of f near ∞, reflected in properties of the class of pairs $(z, f(z))$ for large $|z|$. In fact, the association of the value $f(z)$ with z may, by the substitution of j^{-1}, completely change its character. For instance, the derivative of j near ∞ is the constant[1] function *1*, whereas that of j^{-1} near 0 goes even faster to ∞ than does j^{-1}.

As a result of their emphasis on the mere functional range near ∞, analysts had to assign to ∞ a role differing from that of other points on the Riemann sphere, R, much more profoundly than the arithmetical peculiarity of ∞ seems to warrant. In what follows, the function-theoretical discrepancies between ∞ and the other points on R will be reduced to a minimum. Inevitable are only differences which result from the fact that quotients of the form $(f(z) - f(\infty))/(z - \infty)$ remain undefined, which, in turn, is a direct consequence of the arithmetical peculiarity of ∞, expressed in the formula $a + \infty = \infty$ for any $a \neq \infty$.

Let f be a function whose domain is an open subset of R and whose values belong to R. For any point, a, in Dom f or on its boundary ($a \neq \infty$ or $= \infty$), I will define the *exterior derivative* of f at a as the element b of R (if there is such an element) which has the property

$$\lim [(f(z_n') - f(z_n))/(z_n' - z_n)] = b$$

for any two sequences $\{z_n\}$ and $\{z_n'\}$ of elements $\neq a$ and $\neq \infty$ for which $\lim z_n = a$ and $\lim (z_n' - z_n) = 0$. The number b, which is a complex analogue of the slope of a "paratingent" in the sense of Bouligand,[2] will be denoted by $\mathbf{E}f(a)$. More generally,

$$\mathbf{E}^k f(a) = \lim [(\mathbf{E}^{k-1} f(z_n') - \mathbf{E}^{k-1} f(z_n))/(z_n' - z_n)],$$

where $\mathbf{E}^0 f = f$.

If a is in Dom f and $\neq \infty$, then $\mathbf{E}^k f(a)$ exists and is equal to the classical k-th-order derivative $\mathbf{D}^k f(a)$ if and only if f is regular at a (*i.e.*, differentiable at a and in a neighborhood of a). At ∞, one finds, for instance, $\mathbf{E}j(\infty) = 1$, $\mathbf{E}j^2(\infty) = \infty$, $\mathbf{E}^n j^n(\infty) = n!$ for any positive integer n, and $\mathbf{E}^k j^n(\infty) = 0$ for any negative integer n. If $f(z) = (az + b)/(cz + d)$, where $ad - bc \neq 0$, then $\mathbf{E}f(\infty) = 0$ or a/d, according to whether $c \neq 0$ or $= 0$. Except for their ranges, these functions "behave" at ∞ as they "behave" at 0.

If a point a (finite or ∞) belongs to Dom f or to its boundary, then it will be called a *point of extensibility* of f if $\lim_a f$ exists on R. I will say that, at such a point, f is *steep* if $\mathbf{E}^k f(a) = \infty$ for almost all $k > 0$ and *flat* if $\mathbf{E}^k f(a) = 0$ for almost all

$k > 0$ without $\mathbf{E}^k f$ being constant. Points of extensibility at which f is neither steep nor flat may be called *ordinary;* and f will be said to be ordinary in an open set if and only if the latter consists of ordinary points of f.

If the point a belongs to Dom f or is an isolated boundary point, then f is steep at a if and only if $a \neq \infty$ and a is a pole of f, whereas f is flat at a if and only if $a = \infty$ and f is not a polynomial in the neighborhood of ∞. If Dom f is a wedge containing the real axis and possessing a sufficiently pointed cusp at 0, and if f assumes the value e^{-1/z^2} for any z in that wedge, then 0 is a nonisolated boundary point at which f is flat. If Dom f is the region between the real axis and a curve having this axis as an asymptote of sufficiently high order, and if f assumes the value $sin \sqrt{z}$ for any z in that region, then $\mathbf{E}f(\infty) = 0$, even though ∞ is a point of inextensibility of f.

In many celebrated theorems, the traditional distinction between ∞ and finite points on R, as well as between the interior and the exterior of simple closed curves, becomes unnecessary. For instance, Cauchy's integral theorem can be formulated as follows: If f is ordinary in either complementary region of a rectifiable, simple closed curve Γ and satisfies proper conditions on Γ, then $\int_\Gamma f = 0$. (For, if f is ordinary in the complementary region containing ∞, then f is a polynomial in that region.) Similarly, Cauchy's integral formula follows from

$$\int_\Gamma \frac{f(z) - f(a)}{z - a}\, dz = 0; \qquad \text{briefly,} \qquad \int_\Gamma \frac{f - fa}{j - ja} = 0,$$

for any point a not on Γ and $\neq \infty$.

Suppose that a is an isolated point on the boundary of Dom f, and let Γ be a curve such that the complementary region of Γ containing a consists, with the possible exception of a, of ordinary points of f. Define the residue of f at a as $(1/2\pi i) \int_\Gamma f$, where Γ is traversed once in such a way that the domain containing a is on the left side of Γ. Then, regardless of whether $a \neq \infty$ or $= \infty$, one finds $\mathrm{Res}_a f = 2\pi i c_{-1}$, where c_{-1} is the coefficient of z^{-1} in that Laurent expansion of $f(z)$ about a which is "closest" to a. An expansion about an isolated singularity at ∞ that is closest to ∞ is, of course, a series $\sum_{-\infty}^{\infty} c_n(z - a)^n$ that converges outside some circle (punctured at ∞) about some center a.

If $|f|$ is bounded on the sphere R, punctured anywhere, then f is constant. If, in a punctured circle about a, $|f|$ has either an upper bound $<\infty$ or a lower bound >0 and no upper bound, then a is a point of extensibility of f, and the value at a of the extension is $\neq \infty$ in the first case and $= \infty$ in the second.

The ideas here developed can readily be extended to multifunctions.[3]

* Part of the work on this paper was done as part of Project TB 2-0001(888), sponsored by the Office of Ordnance Research. In writing this paper, the author has profited from several suggestions made by Professor R. Struble.

[1] In this paper, numbers (0, 1, i; only exception π) are denoted in roman type; functions (e.g., the constant function *1* and *j*, assuming the values $1(z) = 1$ and $j(z) = z$ for any z), in italics. Correspondingly, roman letters (z, a, . . .) are used as numerical variables and f as a function variable. (Cf. K. Menger, *Calculus: A Modern Approach* [Boston, 1955].)

[2] Bouligand, *Introduction a la géométrie infinitésimale directe* (Paris, 1932).

[3] K. Menger, "A Simple Definition of Analytic Functions and General Multifunctions," these PROCEEDINGS, **40**, 819–821, 1954.

ON THE GENERATION OF THE IRREDUCIBLE REPRESENTATIONS OF THE SYMMETRIC GROUP

By Francis D. Murnaghan

INSTITUTO TECNOLÓGICO DE AERONÁUTICA, SÃO JOSÉ DOS CAMPOS, BRASIL*

Communicated May 3, 1955

We point out in the present note that the various irreducible representations of the symmetric group S_m on m symbols may all be generated by symmetrizing Kronecker powers of one of these irreducible representations. If $p \rightarrow \begin{pmatrix} p_1 \ldots p_m \\ 1 \ldots m \end{pmatrix}$ is any element of S_m, we obtain an m-dimensional representation of S_m by associating with p the permutation matrix $P = \{e_{p_1}, \ldots, e_{p_m}\}$, where e_{pj}, $j = 1, \ldots, m$, is the $m \times 1$ matrix all of whose elements are zero save the jth, which is 1. This is the representation furnished by the permutations of the cosets of S_{m-1} in S_m under left translations of S_m, and its character is the number α_1 of unary cycles in the class of S_m at which the character is being evaluated. It is reducible, being the sum of the identity representation $\Gamma_{(m)}$ of S_m and of the irreducible representation $\Gamma_{(m-1,\,1)}$, whose character is $\alpha_1 - 1$, of S_m. We denote $\Gamma_{(m-1,\,1)}$ simply by Γ and observe that $\Gamma \otimes \{0\} = \Gamma_{(m)}$ and $\Gamma \otimes \{1\} = \Gamma$. The Kronecker square $\Gamma \times \Gamma$ of Γ is the sum of $\Gamma \otimes \{2\}$ and $\Gamma \otimes \{1^2\}$; the Kronecker cube $\Gamma \times \Gamma \times \Gamma$ of Γ is $(\Gamma \otimes \{3\}) + 2(\Gamma \otimes \{21\}) + (\Gamma \otimes \{1^3\})$; the Kronecker fourth power $\Gamma \times \Gamma \times q$ $\Gamma \times \Gamma$ of Γ is $(\Gamma \otimes \{4\}) + 3(\Gamma \otimes \{31\}) + 2(\Gamma \otimes \{2^2\}) + 3(\Gamma \otimes \{21^2\}) + (\Gamma \otimes \{1^4\})$, and so on (the coefficients 1, 2, 1 in the expression for $\Gamma \times \Gamma \times \Gamma$ being the dimensions of the irreducible representations of S_3, and the coefficients 1, 3, 2, 3, 1 in the expression for $\Gamma \times \Gamma \times \Gamma \times \Gamma$ being the dimensions of the irreducible representations of S_4, and so on). The principal result which we wish to communicate in the present note is that $\Gamma \otimes \{1^k\}$, $k = 1, 2, \ldots, m-1$, is the irreducible representation of S_m whose characteristic is $\{m-k, 1^k\}$ and whose dimension is the binomial coefficient $\binom{m-1}{k}$. If we agree that $\{1^0\} = \{0\}$, we see that S_m has m irreducible representations whose dimensions are the binomial coefficients corresponding to the exponent $m-1$; for $m = 2$ and $m = 3$ these furnish all the irreducible representations of S_m, the dimensions being 1, 1 and 1, 2, 1, respectively. For $m = 4$ they furnish four of the five irreducible representations, namely, those whose dimensions are 1, 3, 3, 1, the remaining irreducible representation being 2-dimensional. We indicate that $\Gamma \otimes \{1^k\}$ is the irreducible representation of S_m whose characteristic is $\{m-k, 1^k\}$ by the notation $\Gamma \otimes \{1^k\} : \{1^k\}$.

It follows readily from the relation $\Gamma \otimes \{1^2\} : \{1^2\}$ that $\Gamma \otimes \{2\} : \{2\} + \{1\} + \{0\}$, i.e., that $\Gamma \otimes \{2\}$ is the sum of the three irreducible representations of S_m whose characteristics are $\{m-2, 2\}$, $\{m-1, 1\}$, and $\{m\}$, respectively. Thus

$$\Gamma_{(m-2,\,2)} = (\Gamma \otimes \{2\}) - (\Gamma \otimes \{1\}) - (\Gamma \otimes \{0\}),$$
$$\Gamma_{(m-2,\,1^2)} = \Gamma \otimes \{1^2\}$$

Similarly, it follows from the relation $\Gamma \otimes \{1^3\} : \{1^3\}$ that

$$\Gamma \otimes \{21\} : \{21\} + \{2\} + \{1^2\} + \{1\},$$
$$\Gamma \otimes \{3\} : \{3\} + \{2\} + \{1^2\} + 2\{1\} + \{0\}.$$

From these and the previous results we read that

$$\Gamma_{(m-3,\,3)} = (\Gamma \otimes \{3\}) - (\Gamma \otimes \{2\}) - (\Gamma \otimes \{1^2\}) - (\Gamma \otimes \{1\}),$$
$$\Gamma_{(m-3,\,21)} = (\Gamma \otimes \{21\}) - (\Gamma \otimes \{2\}) - (\Gamma \otimes \{1^2\}) + (\Gamma \otimes \{0\}),$$
$$\Gamma_{(m-3,\,1^3)} = \Gamma \otimes \{1^3\}.$$

Similarly, it follows from the relation $\Gamma \otimes \{1^4\} : \{1^4\}$ that

$$\Gamma \otimes \{21^2\} : \{21^2\} + \{21\} + \{1^3\} + \{1^2\},$$
$$\Gamma \otimes \{2^2\} : \{2^2\} + \{3\} + \{21\} + 2\{2\} + \{1\} + \{0\},$$
$$\Gamma \otimes \{31\} : \{31\} + \{3\} + 2\{21\} + \{1^3\} + 2\{2\} + 3\{1^2\} + 2\{1\},$$
$$\Gamma \otimes \{4\} : \{4\} + \{3\} + \{21\} + 3\{2\} + \{1^2\} + 3\{1\} + 2\{0\},$$

and from these and the previous results we read that

$$\Gamma_{(m-4,\,4)} = (\Gamma \otimes \{4\}) - (\Gamma \otimes \{3\}) - (\Gamma \otimes \{21\}) - (\Gamma \otimes \{2\}) + (\Gamma \otimes \{1^2\}) + (\Gamma \otimes \{1\}),$$
$$\Gamma_{(m-4,\,31)} = (\Gamma \otimes \{31\} - (\Gamma \otimes \{3\}) - 2(\Gamma \otimes \{21\}) - (\Gamma \otimes \{1^3\}) + (\Gamma \otimes \{2\}) + (\Gamma \otimes \{1\}),$$
$$\Gamma_{(m-4,\,2^2)} = (\Gamma \otimes \{2^2\}) - (\Gamma \otimes \{3\}) - (\Gamma \otimes \{21\}) + 2(\Gamma \otimes \{1^2\}) + 2(\Gamma \otimes \{1\}),$$
$$\Gamma_{(m-4,\,21^2)} = (\Gamma \otimes \{21^2\}) - (\Gamma \otimes \{21\}) - (\Gamma \otimes \{1^3\}) + (\Gamma \otimes \{2\}) - (\Gamma \otimes \{0\}),$$
$$\Gamma_{(m-4,\,1^4)} = \Gamma \otimes \{1^4\},$$

and so on. We call attention to the fact that the right-hand sides of these equations are, formally, independent of m; however, m enters them through $\Gamma = \Gamma_{(m-1,\,1)}$.

* Present address: 6202 Sycamore Road, Baltimore 12, Md.

ON THE ANALYSIS OF THE KRONECKER PRODUCT OF IRREDUCIBLE REPRESENTATIONS OF S_n

BY FRANCIS D. MURNAGHAN

INSTITUTO TECNOLÓGICO DE AERONÁUTICA, SÃO JOSÉ DOS CAMPOS, BRASIL*

Communicated May 25, 1955

We considered,[1] some seventeen years ago, the problem of analyzing the Kronecker product of any two irreducible representations of S_n, the symmetric group on n symbols, into its irreducible components and have given formulas furnishing $\Gamma(n - p, (\mu)) \times \Gamma(n - q, (\nu))$, where (μ) and (ν) are partitions of p and q, respectively, for $p + q \leq 6$ and for $p = 3$, $q = 4$. Furthermore, we availed ourselves of these formulas to furnish tables giving the analysis of the Kronecker product of any two irreducible representations of S_n for $n \leq 8$. Recent papers by Gamba and Radicati[2] and by Robinson and Taulbee[3] have again directed our attention to this question. If p is any nonnegative integer $<n$ and $(\mu_1, \ldots, \mu_j)$ is a partition of p for which $\mu_1 \leq n - p$, we denote simply by (μ) the irreducible representation $\Gamma(n - p, (\mu))$ of S_n which is associated with the partition $(n - p, (\mu))$ of n, and we say that the irreducible representation (μ) of S_n is of *depth p;* thus the identity representation (0) is the one and only irreducible representation of S_n of depth 0,

and $(1) = \Gamma(n-1, 1)$ is the one and only irreducible representation of S_n of depth 1. If (μ) and (ν) are two irreducible representations of S_n of depths p and $q \geq p$, respectively, the greatest depth of the irreducible components of their Kronecker product $(\mu) \times (\nu)$ is $p + q$, and the least depth of these irreducible components $\geq q - p$. Denoting, as usual, by $\{\lambda\}$ the character of the irreducible representation of the full linear group which is associated with the partition $(\lambda_1, \ldots, \lambda_k)$ of any nonnegative integer m, $\{\lambda\}\ \{\lambda'\}$ is a linear combination, with nonnegative integral coefficients, of characters $\{\lambda''\}$, where $(\lambda_1'', \lambda_2'', \ldots)$ is a partition of $m + m'$, and we denote by $(\{\lambda\}\ \{\lambda'\})$ the same linear combination of the irreducible representations (λ''), of depth $m + m'$, of S_n. For example, since $\{1\}\ \{\nu\}$, where $(\nu_1, \ldots, \nu_k)$ is a partition of q, $= \{\nu_1 + 1, \nu_2, \ldots, \nu_k\} + \ldots + \{\nu_1, \ldots, \nu_k + 1\} + \{\nu_1, \ldots, \nu_k, 1\}$, $(\{1\}\ \{\nu\})$ is the linear combination $(\nu_1 + 1, \ldots, \nu_k) + \ldots + (\nu_1, \ldots, \nu_k + 1) + (\nu_1, \ldots, \nu_k, 1)$ of irreducible representations, of depth $q + 1$, of S_n. The first result which we announce in the present note is as follows:

The irreducible components of maximum depth, namely $p + q$, of $(\mu) \times (\nu)$ are furnished by the expression $(\{\mu\}\ \{\nu\})$. For example, the irreducible components of depth 2 of $(1) \times (1)$ are $(2) + (1^2)$. This result, for the special cases $p = 1, 2, 3$, is implicit in formulas given by Makar.[4]

If the partition $(\lambda_1, \ldots, \lambda_k)$ of any nonnegative integer m has k nonzero parts, $\{\lambda\}$ may be written as the determinant of a k-dimensional matrix whose elements are the complete symmetric functions p_j of the characteristic numbers of a typical element of the full linear group, the diagonal elements of the matrix being $p_{\lambda_1}, \ldots, p_{\lambda_k}$. If $(\lambda_1', \ldots, \lambda_{k'}')$ is any partition of any nonnegative integer m' into k' nonzero parts, we understand by $\{\lambda/\lambda'\}$ the determinant of the k-dimensional matrix obtained by reducing the subscripts of the p's in the first, second, and so on, columns of the matrix whose determinant is $\{\lambda\}$ by $\lambda_1', \lambda_2', \ldots$, respectively (it being understood that $\{\lambda/\lambda'\} = 0$ if $k' > k$). It follows readily that $\{\lambda/\lambda'\} = 0$ if $m' > m$ or if, when $m' \leq m$, $\lambda_1' > \lambda_1$; moreover, if $\lambda_1' = \lambda_1$, $\{\lambda/\lambda'\} = \{(\lambda_2, \ldots, \lambda_k)/(\lambda_2', \ldots, \lambda_{k'}')\}$ and so on. It is easy to see that $\{\lambda/1\} = \{\lambda_1 - 1, \lambda_2, \ldots, \lambda_k\} + \ldots + \{\lambda_1, \ldots, \lambda_k - 1\}$, and we understand by $(\{\lambda/1\})$ the sum of the irreducible representations $(\lambda_1 - 1, \ldots, \lambda_k), \ldots, (\lambda_1, \ldots, \lambda_k - 1)$, of depth $m - 1$, of S_n. Similarly, we understand by $(\{\lambda/1\}\ \{1\})$ the sum of irreducible representations, of depth m, of S_n which is furnished by the expression $(\{\lambda_1 - 1, \ldots, \lambda_k\}\ \{1\} + \ldots + \{\lambda_1, \ldots, \lambda_k - 1\}\ \{1\})$. In this notation the second result which we here announce is as follows:

The irreducible components of depth $p + q - 1$ of $(\mu) \times (\nu)$ are furnished by the expression $(\{\mu/1\}\ \{\nu/1\}\ \{1\})$. For example, the irreducible components, of depth q, of $(1) \times (\nu)$ are furnished by the expression $(\{\nu/1\}\ \{1\})$; in particular, the irreducible component, of depth 1, of $(1) \times (1)$ is (1).

Our next result is as follows:

The irreducible components, of depth $p + q - 2$, of $(\mu) \times (\nu)$ are furnished by the expression $(\{\mu/1\}\ \{\nu/1\} + [\{\mu/2\}\ \{\nu/2\} + \{\mu/1^2\}\ \{\nu/1^2\}]\ \{2\} + [\{\mu/2\}\ \{\nu/1^2\} + \{\mu/1^2\}\ \{\nu/2\}]\ \{1^2\})$.

In particular, the irreducible components, of depth $q - 1$, of $(1) \times (\nu)$ are furnished by the expression $(\{\nu/1\})$, since both $\{1/2\}$ and $\{1/1^2\}$ vanish. Since $(1) \times (\nu)$ has no irreducible component of depth $< q - 1$, this completes the analysis of $(1) \times (\nu)$ for all irreducible representations (ν) of S_n:

$$(1) \times (\nu) = (\{\nu/1\} + \{\nu/1\}\{1\} + \{1\}\{\nu\}).$$

For example, $(1) \times (1) = (0) + (1) + (2) + (1^2)$; $(1) \times (2) = (1) + (2) + (1^2) + (3) + (21)$; $(1) \times (1^2) = (1) + (2) + (1^2) + (21) + (1^3)$, and so on. Each of these is a *universal formula* valid for every n; if we evaluate $(1) \times (1^2)$, for example, for $n = 4$ the term $(21) = \Gamma(n - 3, 21)$ vanishes, and we obtain the analysis $\Gamma(31) \times \Gamma(21^2) = \Gamma(31) + \Gamma(2^2) + \Gamma(21^2) + \Gamma(1^4)$.

We content ourselves with giving the formula which furnishes the irreducible components of depth $p + q - 3$ in $(\mu) \times (\nu)$ as the formulas as the depth diminishes. It is

$$([\{\mu/2\} + \{\mu/1^2\}][\{\nu/2\} + \{\nu/1^2\}]\{1\} + [\{\mu/3\}\{\nu/3\} + \{\mu/21\}\{\nu/21\} + \{\mu/1^3\}\{\nu/1^3\}]\{3\} + [\{\mu/3\}\{\nu/21\} + \{\mu/21\}\{\nu/3\} + \{\mu/1^3\}\{\nu/21\} + \{\mu/21\}\{\nu/1^3\}]\{21\} + [\{\mu/3\}\{\nu/1^3\} + \{\mu/1^3\}\{\nu/3\}]\{1^3\}).$$

If (μ) is (2) or (1^2), this reduces to $([\{\nu/2\} + \{\nu/1^2\}]\{1\})$, since the three expressions $\{\mu/3\}, \{\mu/21\}, \{\mu/1^3\}$ vanish. We complete the analysis of $\{\mu\} \times \{\nu\}$ when (μ) is (2) or (1^2) with the observation that the irreducible components of depth $q - 2$ in $(2) \times (\nu)$ are furnished by the expression $(\{\nu/2\})$, and the irreducible components of depth $q - 2$ in $(1^2) \times (\nu)$ are furnished by the expression $(\{\nu/1^2\})$; thus

$$(2) \times (\nu) = (\{\nu/2\} + [\{\nu/2\} + \{\nu/1^2\}]\{1\} + \{\nu/1\}\{1\} + \{\nu/2\}\{2\} + \{\nu/1^2\}\{1^2\} + \{\nu/1\}\{1\}^2 + \{\nu\}\{2\}),$$

$$(1^2) \times (\nu) = (\{\nu/1^2\} + [\{\nu/2\} + \{\nu/1^2\}]\{1\} + \{\nu/1\}\{1\} + \{\nu/1^2\}\{2\} + \{\nu/2\}\{1^2\} + \{\nu/1\}\{1\}^2 + \{\nu\}\{1^2\}).$$

When (μ) is of depth >2, we think it simplest to proceed by a method which we illustrate by the example $(1^3) \times (6)$, whose analysis, in the case $n = 15$, is given by Makar.[4] The terms of depth 9 are $(71^2) + (61^3)$, since $\{1^3\}\{6\} = \{71^2\} + \{61^3\}$; the terms of depth 8 are $(71) + (62) + 2(61^2) + (521) + (51^3)$, since $\{1^3/1\} = \{1^2\}$, $\{6/1\} = \{5\}$, and $\{1^2\}\{5\}\{1\} = [\{61\} + \{51^2\}]\{1\} = \{71\} + \{62\} + 2\{61^2\} + \{521\} + \{51^3\}$; the terms of depth 7 are $2(61) + (52) + 3(51^2) + (421) + (41^3)$, since $\{1^3/2\} = 0$, $\{1^3/1^2\} = \{1\}$, $\{6/2\} = \{4\}$, $\{6/1^2\} = 0$, and $\{1^2\}\{5\} + \{1\}\{4\}\{1^2\} = \{61\} + \{51^2\} + \{1\}[\{51\} + \{41^2\}] = 2\{61\} + \{52\} + 3\{51^2\} + \{42^2\} + \{41^3\}$; the terms of depth 6 are $(6) + 2(51) + (42) + 2(41^2) + (31^3)$, since $\{1^3/3\}$ and $\{1^3/21\}$ are both zero, while $\{1^3/1^3\}$ is the character $\{0\}$ of the identity representation of the full linear group and $\{1\}\{4\}\{1\} + \{3\}\{1^3\} = \{1\}[\{5\} + \{41\}] + \{41^2\} + \{31^3\} = \{6\} + 2\{51\} + \{42\} + 2\{41^2\} + \{31^3\}$. $(1^3) \times (6)$ does not contain any irreducible representation of depth 3, since none of the products $\{1^3\}\{3\}$, $\{1^3\}\{21\}$, $\{1^3\}\{1^3\}$ contains $\{6\}$; similarly, $(1^3) \times (6)$ does not contain any irreducible representation of depth 4, since none of the products $\{1^2\}\{3\}\{1\}$, $\{1^2\}\{21\}\{1\}$, $\{1^2\}\{1^3\}\{1\}$ contains $\{6\}$. We see, in the same way, that the only irreducible representations of depth 5 which can appear in $(1^3) \times (6)$ are (41) and (31^2), and so we know that

$$(1^3) \times (6) = c_{41}(41) + c_{31^2}(31^2) + (6) + 2(51) + (42) + 2(41^2) + (31^3) + 2(61) + (52) + 3(51^2) + (421) + (41^3) + (71) + (62) + 2(61^2) + (521) + (51^3) + (71^2) + (61^3).$$

Setting $n = 2$, we obtain $c_{31^2} = 1$, since $(6) = \Gamma(-4, 6)$ vanishes, as do all the terms on the right save (31^2) and (31^3), which are $-\Gamma(2)$ and $\Gamma(2)$, respectively. Setting $n = 3$, we obtain, similarly, $c_{41} = 1$.

* Present address: 6202 Sycamore Road, Baltimore 12, Md.

[1] F. D. Murnaghan, "The Analysis of the Kronecker Product of Irreducible Representations of the Symmetric Group," *Am. J. Math.*, **60**, 761–784, 1938.

[2] A. Gamba and L. A. Radicati, "Sopra un teorema per la riduzione di talune rappresentazioni del gruppo simmetrico," *Atti accad. nazl. Lincei, Rend. Classe sci. fis. mat. e nat.*, **14**, 632–634, 1953.

[3] G. de B. Robinson and O. E. Taulbee, "The Reduction of the Inner Product of Two Irreducible Representations of S_n," these PROCEEDINGS, **40**, 723–726, 1954.

[4] Ragy H. Makar, "On the Analysis of the Kronecker Product of Irreducible Representations of the Symmetric Group," *Proc. Edinburgh Math. Soc.*, **8**, 133–137, 1949.

---

*AN APPLICATION OF ALGEBRAIC TOPOLOGY TO NUMERICAL ANALYSIS: ON THE EXISTENCE OF A SOLUTION TO THE NETWORK PROBLEM**

BY J. P. ROTH

UNIVERSITY OF CALIFORNIA, BERKELEY

Communicated by G. C. Evans, May 18, 1955

The electrical network problem was first treated comprehensively by Kirchhoff[1] in 1847. A complete proof for the existence of a solution was given by Hermann Weyl[2] in 1923 for the case of a purely resistive network when sources of electromotive force are placed in series with the branches. This proof was elaborated by Eckmann[3] in 1945. A condition for the existence of a solution given by Synge,[4] attempting to cover a more general case, was unfortunately incorrect, as simple counterexamples show. In this paper we show the existence and uniqueness of a solution to the network problem under a condition which amounts to assuming that the dissipative power is positive definite. Since this condition is satisfied by any physically realizable network, it may be said that this result covers the general physical case (steady state). Actually, we state the electrical network problem in a purely algebraic-topological way. This is of interest since "electrical" networks are used to solve certain ordinary and partial differential equations. The results of this paper are essential to the proof of the validity of Kron's method of tearing,[5] established in a second paper.[6] Our first task is to describe a mathematical model for the quantities and relations which exist in an electrical network.

Let K be an electrical network: We shall consider K as an oriented one-dimensional complex. A set of currents flowing through the branches of K may be considered as the assigning of a complex number to each branch ("coil," in Kron's terminology). Hence such a set of currents will be treated as a vector or oriented 1-chain. The space of such sets of branch currents thus coincides with the group $C^1(K)$ of oriented 1-chains over the coefficient field of complex numbers. A mesh current, the current flowing around an oriented closed loop, corresponds to a 1-cycle, but, as we shall see, it is more appropriate to identify it with an element of the first homology group $H^1(K)$ of K. In fact, the space of such loop currents

(they are also called "mesh currents") is isomorphic with $H^1(K)$. With these identifications, the transformation $i = Ci'$ used by Kron[7] is the natural homomorphism[8] of $H^1(K)$ into $C^1(K)$. Here i', i are column vectors representing elements of $H^1(K)$, $C^1(K)$ with respect to bases chosen for these spaces. We recall, also, that $C^1(K)$ may be regarded as the relative homology group of K modulo its zero skeleton. Physically this transformation may be considered as an expression of Kirchhoff's current law. It can also be seen that the space of node currents (node = vertex) is isomorphic to the group $P^0(K)$ of bounding zero chains. In Kron's notation the boundary operator is expressed by the equation $I' = A_t J$, where J and I' are column vectors representing vectors in $C^1(K)$ and $P^0(K)$ with respect to bases chosen for these spaces. We shall use the notation more customary in topology, $I' = \partial J$, ∂ being the boundary operator. In this context the boundary operator may be interpreted as an alternative expression of Kirchhoff's current law, equivalent to the above statement of the same law. We may combine the mappings C and ∂ to form the following "reduced" homology sequence of the pair (K, K^0), K^0 being the zero skeleton of K:

$$0 \to H^1(K) \xrightarrow{C} C^1(K) \xrightarrow{\partial} P^0(K) \to 0.$$

The result $\partial C = 0$ (or $A_t C = 0$) is a consequence of the exactness of this sequence and is referred to by Kron as "the orthogonality condition." Of course this result can also be proved directly very easily.

The dual set of relations involved in considering the spaces of voltages and potentials will next be identified with the corresponding "reduced" cohomology sequence. We start at the opposite end of the sequence. First we identify the space of node potentials with the group $C_0(K)$ of zero cochains of K. The space of potential differences (node-pair potentials) coincides with the subgroup $P_0(K)$ of $C_0(K)$, dual to $P^0(K)$, isomorphic to $C_0(K)$ modulo the subgroup of 0-cocycles. $P_0(K)$ is selected as the image of $P^0(K)$ under the following isomorphism: Let $C^0 = \sum a_i \sigma_i^0$ be an oriented 0-chain of K, the σ_i^0 being oriented 0-cells of K, the summation running over the set of all 0-cells of K, and the a_i being complex numbers. Let $\varphi(C^0)$ be the 0-cochain f such that $f(\sigma_i) = a_i$. The transformation φ is an isomorphism of $C^0(K)$ on $C_0(K)$. Let $P_0(K)$ be the image of $P^0(K)$ under φ. Also, if $\{b_k\}$ is a basis for $P^0(K)$, we shall call the set $\{\varphi(b_k)\}$ of images the "same" basis for $P_0(K)$. Now the space of branch voltages ("coil voltages" in Kron's terminology) may be identified with the group of 1-cochains of K, $C_1(K)$. The coboundary operator $E = \delta E'$ (or $E = AE'$, in Kron's notation), where E, E' represent elements of $C_1(K)$, $P_0(K)$, can be considered as a statement of Kirchhoff's voltage law. Finally, the space of mesh emfs coincides with the first cohomology group $H_1(K)$. Again, if we pick fixed bases for $H^1(K)$ and $C^1(K)$ and write a matrix equation for the natural homomorphism C and then use the "same" basis to express the dual homomorphism of $C_1(K)$ onto $H_1(K)$, then the mapping takes the form used by Kron, $C_t V = \acute{e}$. This mapping is another equivalent statement of Kirchhoff's voltage law. These two mappings yield for us the following reduced cohomology sequence:

$$0 \leftarrow H_1(K) \xleftarrow{C_t} C_1(K) \xleftarrow{\delta} P_0(K) \leftarrow 0.$$

By exactness, $C_t \delta = 0$ (or $C_t A = 0$); this condition is termed by Kron "the orthogonality of mesh emfs and of node-pair potentials."

The Twisted Isomorphism.—When steady-state conditions prevail, the differential equations describing the various branch currents and branch voltages in the network may be reduced to a matrix equation $V = LJ$, a single equation for the entire network, where J is a column vector representing a 1-chain with respect to a basis chosen for $C^1(K)$ and V is a column vector representing a 1-cochain with respect to the "same" basis for $C_1(K)$. Finally, L, called the "impedance matrix," is determined solely by the electromagnetic properties of the various coils (their resistances, mutual inductances, etc.); it is essential to realize that the transformation L is independent of the manner in which the branches are hooked together, i.e., independent of the topological structure of K. In Kron's notation L is denoted by Z.

The electrical network problem may be stated in physical terms as follows: Given a set of coils with given electromagnetic properties determining a transformation L, hooked together in a prescribed way to determine a network K and, given a set e' of sources of emf impressed in series with the meshes (e' is an element of $H_1(K)$) and a set I' of currents impressed from outside the system on the nodes of K (I' is an element of $P^0(K)$), the problem is to find the current flowing through and the potential drop across each coil—these subject to Kirchhoff's laws.

The "electrical" network problem may be stated in purely mathematical terms as follows: Given a complex K (determining the transformations C and ∂) and the matrix L, given $e' \in H_1(K)$, $I' \in P^0(K)$, the problem is to find $V \in C_1(K)$ and $J \in C^1(K)$ such that $1^1\ V = LJ$, $2^0\ C_tV = e'$ and $3^\circ\ \partial J = I'$.

Whether or not a solution to the network problem exists depends, of course, upon the nature of L. We shall say that a matrix L and the transformation represented by L is *power definite* if $L + \bar{L}_t$ is positive definite. The dissipative power $1/2(V_t\bar{J} + \bar{V}_tJ)$ is positive definite if and only if L is power definite. Hence, for any physically realizable network, this condition is satisfied.

THEOREM. *If L is power definite, then the network problem has one and only one solution.*

The problem may be readily visualized by means of the following "algebraic network diagram":

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(K) & \xrightarrow{C} & C^1(K) & \xrightarrow{\partial} & P^0(K) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle L'} & & \downarrow{\scriptstyle L} \quad \uparrow{\scriptstyle Y} & & \uparrow{\scriptstyle Y'} & & \\ 0 & \longleftarrow & H_1(K) & \xleftarrow{C_t} & C_1(K) & \xleftarrow{\delta} & P_0(K) & \longleftarrow & 0 \end{array}$$

Here Y is the inverse of L, $L' = C_tLC$, $Y' = \partial Y\delta$. We will show that under the conditions on L the mappings L' and Y' are one to one, that is, L' and Y' are nonsingular. We will then exhibit the solution. For suppose that $L'i' = 0$, and let $i = Ci'$. Then $\bar{i}_t'L'i'$, which equals $\bar{i}_tLi$, is zero, and so, too, is $\bar{i}_t\bar{L}_t\bar{i}$ zero. Thus $\bar{i}_t(L + \bar{L}_t)i = 0$, and so, by the power-definiteness assumption on L, i must be zero. But since, by exactness, C is an isomorphism into, i' must also be zero. Thus the kernel of L' is zero, and, since the spaces have the same dimension, L' must be one to one. By a similar argument, Y' is also shown to have an inverse. (By exactly the same argument one may show that L' and Y' have inverses if $L - \bar{L}_t$ is assumed to be definite.)

One may easily verify that $J = C(L')^{-1}e' + Y\delta(Y')^{-1}I'$ and $V = LJ$ will constitute a solution. Next we consider the question of uniqueness.

Suppose, then, that V, J and V^*, J^* are solutions to the given problem. Let $V\# = V - V^*$ and $J\# = J - J^*$. Then $\partial J\# = 0$, and so, by the exactness of the homology sequence, there is an $i\#$ in $H^1(K)$ such that $J\# = Ci\#$. But $V\# = LJ\#$, and, since $C_tV\# = 0$, we have $C_tLCi\# = 0$. By the above proof, $i\#$ and hence $J\#$ must be zero, so that $J = J^*$ and $V = V^*$.

Of course, to solve the network problem, to obtain the general solution, it is not necessary to invert both L' and Y'. To solve, for example, by means of inverting L' only, we proceed as follows: Let $C^1(K)$ be written as the direct sum of the image $M^1(K)$ of $H^1(K)$ under C ($M^1(K)$ is the group of cycles) plus another group $P^1(K)$ (cf. companion paper,[6] Sec. II). Then J, an element of $C^1(K)$, can be written as $J = i + I$, $i \in M^1(K)$, $I \in P^1(K)$. Thus I is chosen such that $\partial I = I'$ and may be considered as known. Then $V = L(i + I)$. Now $e' = C_tV = C_tL(i + I)$. But i may be written as the image of an element of $H^1(K)$, $i = Ci'$, $i' \in H^1(K)$. Thus $C_tLCi' = e' - C_tLI$, so that $i' = (L')^{-1}(e' - C_tLI)$. Thus J is determined, $J = Ci' + I$, and $V = LJ$, so that the solution is obtained by inverting L' only. Similarly, one may obtain the solution by inverting Y'. We also have the following consequence of the above theorem.

COROLLARY. *If a network problem is physically realizable, then there exists one and only one solution.*

* Presented before the American Mathematical Society, December 27, 1954, under the title "An Application of Algebraic Topology to Numerical Analysis: Kron's Method I."

[1] G. Kirchhoff, "Ueber die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird," Poggendorf's Ann. Physik Chemie, **72**, 497–508, 1847.

[2] H. Weyl, "Repartición de corriente en uno red conductora," *Rev. mat. hispano-americana*, **5**, 153–164, 1923.

[3] B. Eckmann, "Harmonische Funktionen und Randwertaufgaben in einem Komplex," *Comm. Math. Helv.*, **17**, 240–255, 1944–1945.

[4] J. L. Synge, "The Fundamental Theorem of Electrical Networks," *Quart. Appl. Math.*, **9**, 113–127, 1951.

[5] G. Kron, "A Set of Principles To Interconnect the Solutions of Physical Systems," *J. Appl Phys.*, **24**, 965–980, 1953.

[6] J. P. Roth, "The Validity of Kron's Method of Tearing," these PROCEEDINGS **41**, 1955, in press.

[7] G. Kron, *Tensor Analysis of Networks* (New York: John Wiley & Sons, Inc., 1939).

[8] S. Eilenberg and N. E. Steenrod, *Foundations of Algebraic Topology* (Princeton, N. J.: Princeton University Press, 1952).

THE ROLES OF THE OPTIC VESICLE AND OTHER HEAD TISSUES IN LENS INDUCTION

By Antone G. Jacobson

DEPARTMENT OF BIOLOGICAL SCIENCES, STANFORD UNIVERSITY

Communicated by V. C. Twitty, May 12, 1955

The induction of the amphibian lens, although one of the most extensively investigated phenomena in amphibian development, has continued to present perplexing if not paradoxical features. The early work of Spemann and of Lewis established that the optic vesicle has an active role in lens induction.[1] In some species, however, it was found that lenses could be formed in the absence of the optic vesicle. While Spemann had found this to be the case in *Rana esculenta*,[1] others, notably Woerdeman,[2] were unable to obtain independent lenses in esculenta.

Studying determination of ciliary polarity in embryos of *Amblystoma punctatum*, Twitty[3] observed that low temperature slows the process of determination less than it retards the rate of morphological development. Following this lead, Ten Cate,[4] working in Woerdeman's laboratory, has shown that the difference between Woerdeman's and Spemann's results with esculenta is attributable to the lower temperature at which Spemann kept his embryos prior to the extirpation of the retinal anlage. There still remained the problem of identifying the inductors involved in evoking a lens in the absence of the optic vesicle.

Liedke[5] has demonstrated in *A. punctatum* that "lens competence is acquired during the open neural fold stages through activation by the anterior ento-mesoderm." Considering together the effects of temperature and the role of the "anterior ento-mesoderm," Twitty[6] has suggested that

> . . . probably in all species lens formation is ordinarily elicited by the action of at least two supplementary or mutually reinforcing inductors, the head mesoderm and the retinal rudiment. Under average environmental circumstances and in most species the component events of development are geared with one another in such manner that the epidermis is insufficiently differentiated at the neurula stage to be fully receptive to the action of the mesoderm, and requires a subsequent and final impetus from the optic vesicle. At low temperatures, however, and especially in species like *R. esculenta* in which such treatment causes morphological development to lag substantially behind chemical differentiation, the lens epidermis has already become sufficiently mature and responsive at neurulation to require no further incitement beyond that provided by the head mesoderm.

This suggestion has been substantiated in the present study by combining on a single species, *Triturus torosus*, the temperature and "head mesoderm" experiments and also by the results of isolating from the young neurula presumptive lens epidermis together with the subjacent tissues in vitro.

As shown in Figure 1, the temperature at which torosus embryos are reared prior to excision of the retinal anlage in the open neural plate stage has a pronounced effect on the percentage of animals which develop lenses in the absence of the optic vesicle. An example of an embryo with an independent lens is shown in Figure 2.

Liedke[5] had found in *A. punctatum* that gastrula epidermis transplanted to the lens site develops a lens only if it is in position during neurulation. Lenses developed in the graft if the host was an early neurula but not if the host was a late

neurula. In the present experiments with torosus, this same distinction was evident but not so clear cut. Lenses developed in 71 per cent of the grafts to early neurula hosts and in 28 per cent of the grafts to late neurula hosts.

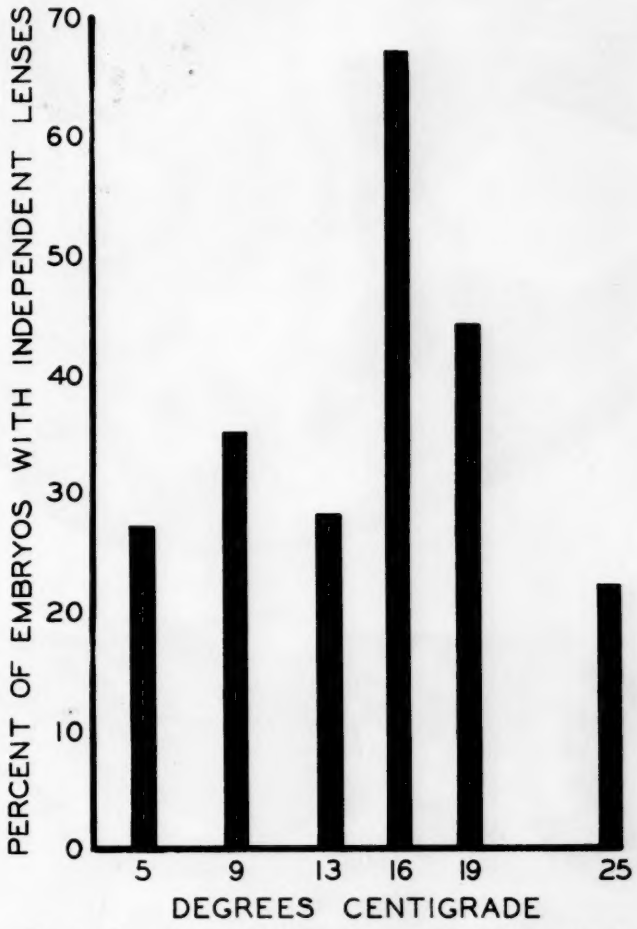
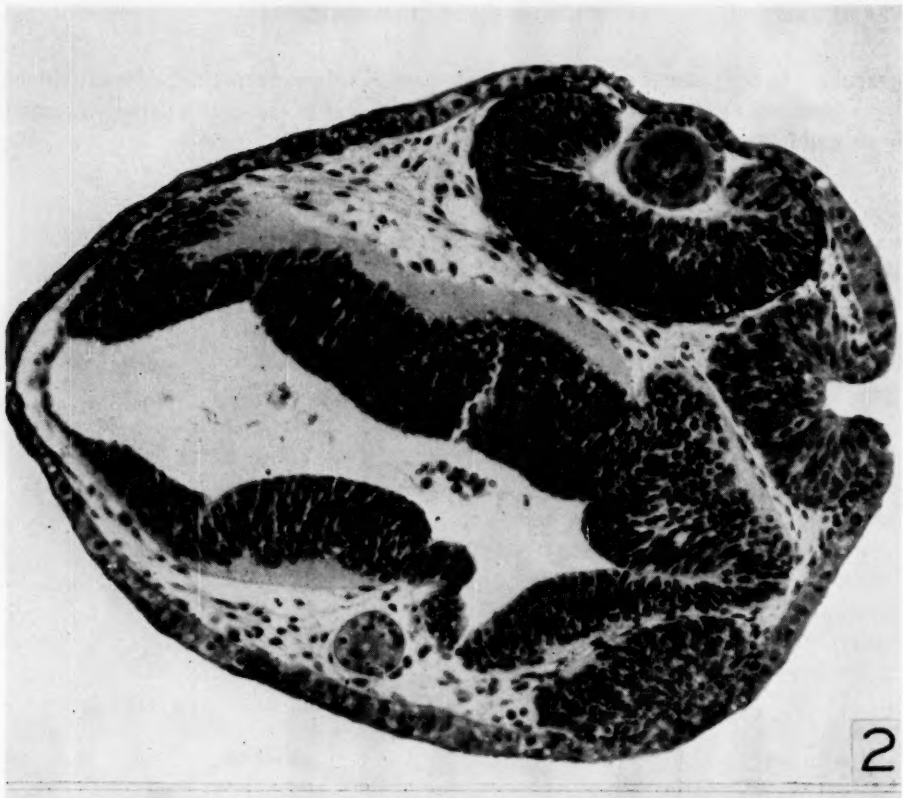


FIG. 1.—Percentage of torosus embryos which develop a lens in the absence of the optic vesicle after rearing the groups of embryos at the indicated temperatures prior to excision of the retinal rudiment in the early neurula (stage 16). All groups were kept at 19° C. after the operation.

In the early neurula the presumptive lens epidermis is underlaid directly by the entodermal wall of the archenteron; the anterior edge of the lateral plate mesoderm lies just posterior to the lens site. Presumptive lens epidermis from young neurulae was isolated in combination with (1) the subjacent entodermal wall of the archenteron; (2) the anterior portion of the lateral plate mesoderm; and (3) these two tissues combined (Fig. 3). Lenses were induced in a very substantial percentage of cases in all three series. No lenses have been found in presumptive lens



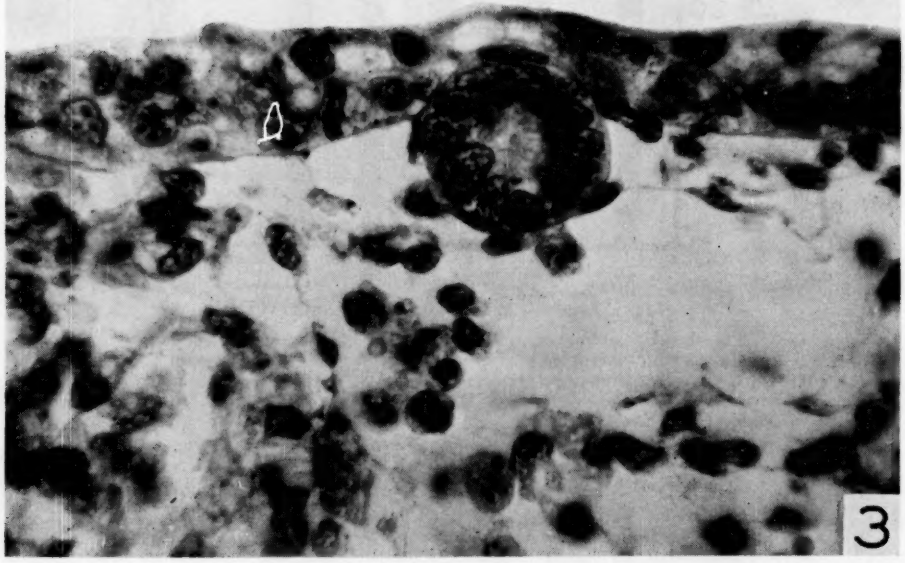


FIG. 2.—Transverse section of a 12-mm. torosus larva in which a lens has developed in the absence of the retina. The embryo was reared at 13° C. until the early neurula stage, when the right retinal anlage was excised.

FIG. 3.—Section of an explant of presumptive lens epidermis combined with the subjacent entodermal wall of the archenteron and the anterior portion of the lateral plate mesoderm; taken from a young torosus neurula (stage 16). A lens has been induced from the epidermis.

epidermis isolated alone. These explants were taken from embryos which had been reared at 9° or at 13° C.

The results of these experiments indicate that lens induction is the result of the synergistic action of the optic vesicle and of the nonneural tissues underlying the presumptive lens epidermis during neurulation. In fact, these latter tissues may by themselves induce lenses, without the action of the optic vesicle, especially in embryos that have been reared at appropriate temperatures. The emergence of the lens under the influence of more than one inductor finds a close parallel in the induction of other head organs, such as the otocyst, the olfactory organ, and the hypophysis.[7]

[1] H. Spemann, *Embryonic Development and Induction* (New Haven: Yale University Press, 1938), chap. iii.

[2] M. W. Woerdeman, *Koninkl. Ned. Akad. Wetenschap., Proc.*, **42**, 290–292, 1939.

[3] V. C. Twitty, *J. Exptl. Zool.*, **50**, 319–344, 1928.

[4] G. Ten Cate, *The Intrinsic Development of Amphibian Embryos* (Amsterdam: North-Holland Publishing Co., 1953), chap. vi.

[5] K. B. Liedke, *J. Exptl. Zool.*, **117**, 573–591, 1951.

[6] V. C. Twitty, in Willier, Weiss, and Hamburger, *Analysis of Development* (Philadelphia & London: W. B. Saunders Co., 1955), pp. 402–414.

[7] J. Holtfreter, *Growth* (*suppl.*), pp. 117–152, 1951.

NATIONAL ACADEMY OF SCIENCES

July 1, 1955

OFFICERS

	Term expires
President—Detlev W. Bronk	June 30, 1958
Vice President—George W. Corner	June 30, 1957
Home Secretary—Hugh L. Dryden	June 30, 1959
Foreign Secretary—John G. Kirkwood	June 30, 1958
Treasurer—William J. Robbins	June 30, 1956

Executive Officer
S. D. Cornell

Business Manager
G. D. Meid

COUNCIL

*Bronk, Detlev W.	(1958)	*Kirkwood, John G.	(1958)
*Corner, George W.	(1957)	Painter, Theophilus S.	(1958)
Daniels, Farrington	(1957)	*Robbins, William J.	(1956)
Doisy, E. A.	(1958)	*Tuve, Merle A.	(1957)
*Dryden, Hugh L.	(1959)	*Wilson, Edwin B.	(1956)
Gilluly, James	(1956)		

MEMBERS

The number in parentheses, following year of election, indicates the Section to which the member belongs, as follows:

(1)	Mathematics	(8)	Zoology and Anatomy
(2)	Astronomy	(9)	Physiology
(3)	Physics	(10)	Pathology and Bacteriology
(4)	Engineering	(11)	Anthropology
(5)	Chemistry	(12)	Psychology
(6)	Geology	(13)	Geophysics
(7)	Botany	(14)	Biochemistry

Abbot, Charles **G**reeley, 1915 (2), Smithsonian Institution, Washington 25, D. C.

Adams, Comfort **A**very, 1930 (4), 417 West Price Street, Philadelphia 44, Pennsylvania

Adams, Leason **H**eberling, 1943 (13), Carnegie Institution of Washington, 1530 P Street, N. W., Washington 5, D. C.

Adams, Roger, 1929 (5), Department of Chemistry and Chemical Engineering, University of Illinois, Urbana, Illinois

Adams, Walter Sydney, 1917 (2), Mount Wilson Observatory, Pasadena 4, California

Ahlfors, Lars Valerian, 1953 (1), Department of Mathematics, Harvard University, Cambridge 38, Massachusetts

Albert, Abraham **Adrian,** 1943 (1), Department of Mathematics, University of Chicago, Chicago 37, Illinois

* Members of the Executive Committee of the Council of the Academy.

Albright, Fuller, 1952 (10), Massachusetts General Hospital, Boston 14, Massachusetts

Albright, William Foxwell, 1955 (11), Oriental Seminary, Johns Hopkins University, Baltimore 18, Maryland

Alexander, James Waddell, 1930 (1), 29 Cleveland Lane, Princeton, New Jersey

Allen, Eugene Thomas, 1930 (6), The Irvington, 135 Pleasant Street, Arlington 74, Massachusetts

Allison, Samuel King, 1946 (3), Institute for Nuclear Studies, University of Chicago, Chicago 37, Illinois

Alvarez, Luis Walter, 1947 (3), Radiation Laboratory, University of California, Berkeley 4, California

Anderson, Carl David, 1938 (3), California Institute of Technology, Pasadena 4, California

Anderson, Edgar, 1954 (7), Missouri Botanical Garden, 2315 Tower Grove Avenue, St. Louis 10, Missouri

Anderson, Rudolph John, 1946 (14), Sterling Chemistry Laboratory, Yale University, 225 Prospect Street, New Haven 11, Connecticut

Armstrong, Charles, 1944 (10), National Institutes of Health, Bethesda 14, Maryland

Babcock, Harold Delos, 1933 (2), 1820 Atchison Street, Pasadena, California

Babcock, Horace Welcome, 1954 (2), Mount Wilson Observatory, 813 Santa Barbara Street, Pasadena 4, California

Bacher, Robert Fox, 1947 (3), California Institute of Technology, Pasadena, California

Badger, Richard McLean, 1952 (5), Gates and Crellin Laboratories of Chemistry, California Institute of Technology, Pasadena 4, California

Bailey, Irving Widmer, 1929 (7), Harvard University Herbarium, 22 Divinity Avenue, Cambridge 38, Massachusetts

Bailey, Percival, 1953 (10), Neuropsychiatric Institute, 912 South Wood Street, Chicago 12, Illinois

Bain, Edgar Collins, 1954 (4), United States Steel Corporation, 525 Wm. Penn Place, Pittsburgh 30, Pennsylvania

Bainbridge, Kenneth Tompkins, 1946 (3), Department of Physics, Harvard University, Cambridge 38, Massachusetts

Ball, Eric Glendinning, 1948 (14), Department of Biological Chemistry, Harvard Medical School, 25 Shattuck Street, Boston 15, Massachusetts

Balls, Arnold Kent, 1954 (14), Department of Biochemistry, Purdue University, Lafayette, Indiana

Bard, Philip, 1944 (9), School of Medicine, Johns Hopkins University, 710 North Washington Street, Baltimore 5, Maryland

Bardeen, John, 1954 (3), Department of Physics, University of Illinois, Urbana, Illinois

Barker, Horace Albert, 1953 (7), Department of Plant Biochemistry, 337 Biochemistry and Virus Laboratory, University of California, Berkeley 4, California

Bartelmez, George William, 1949 (8), Department of Embryology, Carnegie Institution of Washington, Wolfe and Madison Streets, Baltimore 5, Maryland

Bartlett, Paul Doughty, 1947 (5), Department of Chemistry, Harvard University, 12 Oxford Street, Cambridge 38, Massachusetts

Beach, Frank Ambrose, 1949 (12), Department of Psychology, Yale University, 333 Cedar Street, New Haven 11, Connecticut

Beadle, George Wells, 1944 (7), Division of Biology, California Institute of Technology, Pasadena, California

Beams, Jesse Wakefield, 1943 (3), Physical Laboratory, University of Virginia, McCormick Road, Charlottesville, Virginia

Bell, Eric Temple, 1927 (1), 434 South Michigan Avenue, Pasadena 5, California

Benioff, Victor **Hugo,** 1953 (13), Seismological Laboratory, California Institute of Technology, 220 North San Rafael Avenue, Pasadena 2, California

Berkey, Charles Peter, 1927 (6), Department of Geology, Columbia University, New York 27, New York

Berkner, Lloyd Viel, 1948 (13), Associated Universities, Incorporated, 350 Fifth Avenue, New York 1, New York

Bethe, Hans Albrecht, 1944 (3), Laboratory of Nuclear Studies, Cornell University, Ithaca, New York

Bigelow, Henry Bryant, 1931 (8), Museum of Comparative Zoology at Harvard College, Oxford Street, Cambridge 38, Massachusetts

Birch, Albert **Francis,** 1950 (6), Dunbar Laboratory, Harvard University, Cambridge 38, Massachusetts

Birge, Raymond Thayer, 1932 (3), University of California, Berkeley 4, California

Bjerknes, Jacob, 1947 (13), Department of Meteorology, University of California, Los Angeles 24, California

Blackwelder, Eliot, 1936 (6), P. O. Box N, Stanford, California

Blalock, Alfred, 1945 (10), Johns Hopkins Hospital, Baltimore 5, Maryland

Blinks, Lawrence Rogers, 1955 (7), Hopkins Marine Station of Stanford University, Pacific Grove, California

Bloch, Felix, 1948 (3), CERN, Geneva, Switzerland

Bloom, William, 1954 (8), Department of Anatomy, University of Chicago, Chicago 37, Illinois

Bochner, Salomon, 1950 (1), Department of Mathematics, Princeton University, Princeton, New Jersey

Bolton, Elmer **K.,** 1946 (5), 2310 West 11th Street, Wilmington, Delaware

Bonner, James Frederick, 1950 (7), Kerckhoff Laboratories of Biology, California Institute of Technology, Pasadena 4, California

Boring, Edwin Garrigues, 1932 (12), Memorial Hall, Harvard University, Cambridge 38, Massachusetts

Bowen, Ira **Sprague,** 1936 (2), Mount Wilson and Palomar Observatories, 813 Santa Barbara Street, Pasadena 4, California.

Bowen, Norman Levi, 1935 (6), Geophysical Laboratory, Carnegie Institution of Washington, 2801 Upton Street, N. W., Washington 8, D. C.

Bradbury, Norris Edwin, 1951 (3), Los Alamos Scientific Laboratory, P. O. Box 1663, Los Alamos, New Mexico

Bradley, Wilmot **Hyde,** 1946 (6), United States Geological Survey, Department of the Interior, Washington 25, D. C.

Bramlette, Milton **Nunn,** 1954 (6), Scripps Institution of Oceanography, La Jolla, California

Brauer, Richard Dagobert, 1955 (1), Department of Mathematics, Harvard University, Cambridge 38, Massachusetts

Breit, Gregory, 1939 (3), Sloane Physics Laboratory, Yale University, New Haven 11, Connecticut

Bridgman, Percy Williams, 1918 (3), Lyman Laboratory of Physics, Harvard University, Cambridge 38, Massachusetts

Briggs, Lyman James, 1942 (4), National Bureau of Standards, Washington 25, D. C.

Brillouin, Leon, 1953 (3), 88 Central Park West, New York 23, New York

Brink, Royal Alexander, 1947 (7), Department of Genetics, University of Wisconsin, Madison 6, Wisconsin

Brode, Robert Bigham, 1949 (3), Department of Physics, University of California, Berkeley 4, California

Brode, Wallace Reed, 1954 (5), National Bureau of Standards, Washington 25, D. C.

Bronk, Detlev Wulf, 1939 (9), Rockefeller Institute for Medical Research, New York 21, New York

Brouwer, Dirk, 1951 (2), Yale University Observatory, Prospect and Canner Streets, New Haven 11, Connecticut

Brown, Harrison Scott, 1955 (13), Division of Geological Sciences, California Institute of Technology, Pasadena, California

Bucher, Walter Hermann, 1938 (6), Department of Geology, Columbia University, New York 27, New York

Buckley, Oliver Ellsworth, 1937 (4), 13 Fairview Terrace, Maplewood, New Jersey

Buddington, Arthur Francis, 1943 (6), Princeton University, Princeton, New Jersey

Buerger, Martin Julian, 1953 (6), Department of Geology and Geophysics, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Burkholder, Paul Rufus, 1949 (7), Department of Bacteriology, University of Georgia, Athens, Georgia

Burns, Robert Kyle, 1955 (8), Department of Embryology, Carnegie Institution of Washington, Wolfe and Madison Streets, Baltimore 5, Maryland

Bush, Vannevar, 1934 (4), Carnegie Institution of Washington, 1530 P Street, N.W., Washington 5, D. C.

Byerly, Perry, 1946 (13), Department of Geological Sciences, University of California, Berkeley 4, California

Byers, Horace Robert, 1952 (13), Department of Meteorology, University of Chicago, Chicago 37, Illinois

Calvin, Melvin, 1954 (5), Department of Chemistry, University of California, Berkeley 4, California

Cannon, Paul Roberts, 1946 (10), Department of Pathology, University of Chicago, Chicago 37, Illinois

Carlson, Anton Julius, 1920 (9), University of Chicago, Chicago 37, Illinois

Carmichael, Leonard, 1943 (12), Smithsonian Institution, Washington 25, D. C.

Carter, Herbert Edmund, 1953 (14), Department of Chemistry, University of Illinois, Urbana, Illinois

Castle, William Bosworth, 1939 (10), Boston City Hospital, Boston 18, Massachusetts

Castle, William Ernest, 1915 (8), 421 Spruce Street, Berkeley 8, California

Chance, Britton, 1954 (14), Johnson Foundation, University of Pennsylvania, Philadelphia 4, Pennsylvania

Chandler, William Henry, 1943 (7), College of Agriculture, University of California, 405 Hilgard Avenue, Los Angeles 24, California

Chandrasekhar, Subrahmanyan, 1955 (2), Yerkes Observatory, University of Chicago, Williams Bay, Wisconsin

Chaney, Ralph Works, 1947 (6), Department of Paleontology, University of California, Berkeley 4, California

Chevalley, Claude, 1952 (1), 468 Riverside Drive, New York 27, New York

Chipman, John, 1955 (4), Department of Metallurgy, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Clark, William Mansfield, 1928 (14), Department of Chemistry, Johns Hopkins University, Baltimore 18, Maryland

Clarke, Hans Thacher, 1942 (14), College of Physicians and Surgeons, 630 West 168th Street, New York 32, New York

Clausen, Roy Elwood, 1951 (7), 348 Hilgard Hall, University of California, Berkeley 4, California

Cleland, Ralph Erskine, 1942 (7), Department of Botany, Indiana University, Bloomington, Indiana

Clemence, Gerald Maurice, 1952 (2), United States Naval Observatory, Washington 25, D. C.

Cleveland, Lemuel Roscoe, 1952 (8), The Biological Laboratories, Harvard University, 16 Divinity Avenue, Cambridge 38, Massachusetts

Cloos, Ernst, 1950 (6), Department of Geology, Johns Hopkins University, Baltimore 18, Maryland

Coble, Arthur Byron, 1924 (1), 1907 Tondolea Lane, La Canada, California

Coblentz, William Weber, 1930 (3), 2737 Macomb Street, N. W., Washington 8, D. C.

Cochrane, Edward Lull, 1945 (4), Office of Industrial and Governmental Relations, Room 10-200, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Coggeshall, Lowell Thelwell, 1949 (10), Division of Biological Sciences, University of Chicago, Chicago 37, Illinois

Cole, Rufus, 1922 (10), Mt. Kisco, New York

Compton, Arthur Holly, 1927 (3), Washington University, St. Louis 5, Missouri

Conant, James Bryant, 1929 (5), United States Ambassador to the Federal Republic of Germany, A.P.O. 80, Box 100, New York, New York

Condon, Edward Uhler, 1944 (3), 2938 Avalon Avenue, Berkeley 5, California

Coolidge, William David, 1925 (3), 1480 Lenox Road, Schenectady 8, New York

Coon, Carleton Stevens, 1955 (11), The University Museum, University of Pennsylvania, 33rd and Spruce Streets, Philadelphia 4, Pennsylvania

Cope, Arthur Clay, 1947 (5), Department of Chemistry, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Cori, Carl Ferdinand, 1940 (14), School of Medicine, Washington University, Euclid Avenue and Kingshighway, St. Louis 10, Missouri

Cori, Gerty Theresa, 1948 (14), School of Medicine, Washington University, Euclid Avenue and Kingshighway, St. Louis 10, Missouri

Corner, George Washington, 1940 (8), Department of Embryology, Carnegie Institution of Washington, Wolfe and Madison Streets, Baltimore 5, Maryland. After January 1, 1956: Rockefeller Institute for Medical Research, New York 21, New York

Couch, John Nathaniel, 1943 (7), University of North Carolina, Chapel Hill, North Carolina

Courant, Richard, 1955 (1), Institute of Mathematical Sciences, New York University, 25 Waverly Place, New York 3, New York

Craig, Lyman Creighton, 1950 (5), Rockefeller Institute for Medical Research, New York 21, New York

Curme, George Oliver, Jr., 1944 (4), Union Carbide and Carbon Corporation, 30 East 42nd Street, New York 17, New York

Dalldorf, Gilbert, 1955 (10), New York State Department of Health, Albany, New York

Daly, Reginald Aldworth, 1925 (6), 23 Hawthorn Street, Cambridge 38, Massachusetts

Danforth, Charles Haskell, 1942 (8), Department of Anatomy, Stanford University, Stanford, California

Daniels, Farrington, 1947 (5), Department of Chemistry, University of Wisconsin, Madison 6, Wisconsin

Davis, Bergen, 1929 (3), 44 Morningside Drive, New York 25, New York

Davis, Hallowell, 1948 (9), Central Institute for the Deaf, 818 South Kingshighway, St. Louis 10, Missouri

Davisson, Clinton Joseph, 1929 (3), 2605 Jefferson Park Circle, Charlottesville, Virginia

Day, Arthur Louis, 1911 (6), 9113 Old Georgetown Road, Bethesda 14, Maryland

Debye, Peter, (1931) 1947* (5), Baker Laboratory, Cornell University, Ithaca, New York

DeGolyer, Everette Lee, 1951 (6), 5625 Daniels Street, Dallas, Texas

Delbrück, Max, 1949 (7), Kerckhoff Laboratories of Biology, California Institute of Technology, Pasadena 4, California

Demerec, Milislav, 1946 (8), Department of Genetics, Carnegie Institution of Washington, Cold Spring Harbor, Long Island, New York

Den Hartog, Jacob Pieter, 1953 (4), Department of Mechanical Engineering, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Dennison, David Mathias, 1953 (3), Randall Laboratory of Physics, University of Michigan, Ann Arbor, Michigan

Detwiler, Samuel Randall, 1932 (8), College of Physicians and Surgeons, 630 West 168th Street, New York 32, New York

Dobzhansky, Theodosius, 1943 (8), Department of Zoology, Columbia University, New York 27, New York

Dochez, Alphonse Raymond, 1933 (10), Columbia University, Presbyterian Hospital, 620 West 168th Street, New York 32, New York

Dodge, Bernard Ogilvie, 1933 (7), New York Botanical Garden, Bronx Park (Fordham Station), New York 58, New York

* Elected a foreign associate in 1931; became a naturalized citizen in 1946 and a member of the Academy in 1947.

Doisy, Edward Adelbert, 1938 (14), St. Louis University School of Medicine, 1402 South Grand Boulevard, St. Louis 4, Missouri

Douglas, Jesse, 1946 (1), Butler Hall, 400 West 119th Street, New York 27, New York

Dragstedt, Lester Reynold, 1950 (10), Department of Surgery, University of Chicago, 950 East 59th Street, Chicago 37, Illinois

Dryden, Hugh Latimer, 1944 (4), National Advisory Committee for Aeronautics, 1512 H Street, N. W., Washington 25, D. C.

DuBois, Eugene Floyd, 1933 (9), Cornell University Medical College, 1300 York Avenue, New York 21, New York

Dubos, René Jules, 1941 (10), Rockefeller Institute for Medical Research, New York 21, New York

DuBridge, Lee Alvin, 1943 (3), California Institute of Technology, Pasadena 4, California

Duggar, Benjamin Minge, 1927 (7), Lederle Laboratories Division, American Cyanamid Company, Pearl River, New York

DuMond, Jesse William Monroe, 1953 (3), Department of Physics, California Institute of Technology, Pasadena 4, California

Dunbar, Carl Owen, 1944 (6), Peabody Museum, Yale University, New Haven 11, Connecticut

Dunn, Leslie Clarence, 1943 (8), Columbia University, New York 27, New York

Dunning, John Ray, 1948 (3), 301 Engineering Building, Columbia University, New York 27, New York

Durand, William Frederick, 1917 (4), Mohawk Hotel, 379 Washington Avenue, Brooklyn, New York

du Vigneaud, Vincent, 1944 (14), Department of Biochemistry, Cornell University Medical College, 1300 York Avenue, New York 21, New York

Eckart, Carl, 1953 (13), Scripps Institution of Oceanography, La Jolla, California

Edsall, John Tileston, 1951 (5), The Biological Laboratories, Harvard University, 16 Divinity Avenue, Cambridge 38, Massachusetts

Eisenhart, Luther Pfahler, 1922 (1), 25 Alexander Street, Princeton, New Jersey

Elderfield, Robert Cooley, 1949 (5), Department of Chemistry, University of Michigan, Ann Arbor, Michigan

Elvehjem, Conrad Arnold, 1942 (14), Department of Biochemistry, University of Wisconsin, Madison 6, Wisconsin

Emerson, Robert, 1953 (7), Department of Botany, University of Illinois, Urbana, Illinois

Emmett, Paul Hugh, 1955 (5), Department of Chemistry, Johns Hopkins University, Baltimore 18, Maryland

Enders, John Franklin, 1953 (10), The Children's Hospital, 300 Longwood Avenue, Boston 15, Massachusetts

Epstein, Paul Sophus, 1930 (3), 1484 Oakdale Street, Pasadena 4, California

Erlanger, Joseph, 1922 (9), 5127 Waterman Boulevard, St. Louis 8, Missouri

Evans, Griffith Conrad, 1933 (1), Department of Mathematics, University of California, Berkeley 4, California

Evans, Herbert McLean, 1927 (9), Institute of Experimental Biology, University of California, Berkeley 4, California

Ewing, William **Maurice,** 1948 (13), Columbia University, Lamont Geological Observatory, Torrey Cliff, Palisades, New York

Eyring, Henry, 1945 (5), Graduate School, University of Utah, Salt Lake City, Utah

Fenn, Wallace Osgood, 1943 (9), School of Medicine and Dentistry, University of Rochester, 260 Crittenden Boulevard, Rochester 20, New York

Feynman, Richard Phillips, 1954 (3), Norman Bridge Laboratory of Physics, California Institute of Technology, Pasadena 4, California

Fieser, Louis Frederick, 1940 (5), Harvard University, Cambridge 38, Massachusetts

Fischer, Hermann Otto Laurenz, 1954 (14), Department of Biochemistry, University of California, Berkeley 4, California

Fisk, James Brown, 1954 (4), Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey

Fleming, John Adam, 1938 (13), Apt. 6A, Shelby Court, 4th Street and Shelby Avenue, Nashville 6, Tennessee

Fletcher, Harvey, 1935 (4), College of Physical and Engineering Sciences, Brigham Young University, Provo, Utah

Flory, Paul John, 1953 (5), Department of Chemistry, Cornell University, Ithaca, New York

Folkers, Karl August, 1948 (5), Organic and Biological Chemical Research Division, Merck & Co., Inc., Rahway, New Jersey

Foote, Paul Darwin, 1943 (4), P. O. Drawer 2038, Pittsburgh 30, Pennsylvania

Forbes, Alexander, 1936 (9), The Biological Laboratories, Harvard University, 16 Divinity Avenue, Cambridge 38, Massachusetts

Francis, Thomas, Jr., 1948 (10), Department of Epidemiology, School of Public Health, University of Michigan, Ann Arbor, Michigan

Franck, James, 1944 (3), Research Institutes, University of Chicago, 5650 Ellis Avenue, Chicago 37, Illinois

Fred, Edwin Broun, 1931 (7), University of Wisconsin, Madison 6, Wisconsin

Fruton, Joseph Stewart, 1952 (14), Department of Biochemistry, Yale University School of Medicine, 333 Cedar Street, New Haven 11, Connecticut

Fuoss, Raymond Matthew, 1951 (5), Sterling Chemistry Laboratory, Yale University, 225 Prospect Street, New Haven 11, Connecticut

Fuson, Reynold Clayton, 1944 (5), 263 Noyes Laboratory, University of Illinois, Urbana, Illinois

Gamble, James Lawder, 1945 (10), 33 Edge Hill Road, Brookline, Massachusetts

Gamow, George, 1953 (3), Department of Physics, George Washington University, Washington, D. C.

Gasser, Herbert Spencer, 1934 (9), Rockefeller Institute for Medical Research, New York 21, New York

Gerard, Ralph Waldo, 1955 (9), Neuropsychiatric Institute, 912 South Wood Street, Chicago 12, Illinois

Gesell, Arnold, 1947 (12), Gesell Institute of Child Development, 310 Prospect Street, New Haven 11, Connecticut

Giauque, William Francis, 1936 (5), Department of Chemistry and Chemical Engineering, University of California, Berkeley 4, California

Gibbs, William Francis, 1949 (4), One Broadway, New York, New York

Gilliland, Edwin Richard, 1948 (4), Department of Chemical Engineering, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Gilluly, James, 1947 (6), United States Geological Survey, Denver Federal Center, Denver 14, Colorado

Gilman, Henry, 1945 (5), Department of Chemistry, Iowa State College, Ames, Iowa

Goddard, David Rockwell, 1950 (7), Botanical Laboratory, University of Pennsylvania, Philadelphia 4, Pennsylvania

Gödel, Kurt, 1955 (1), The Institute for Advanced Study, Princeton, New Jersey

Goldschmidt, Richard Benedikt, 1947 (8), Department of Zoology, University of California, Berkeley 4, California

Goodpasture, Ernest William, 1937 (10), Armed Forces Institute of Pathology, Walter Reed Army Medical Center, 6825 16th Street, N. W., Washington 25, D. C.

Goudsmit, Samuel Abraham, 1947 (3), Department of Physics, Brookhaven National Laboratory, Upton, Long Island, New York

Graham, Clarence Henry, 1946 (12), Department of Psychology, Columbia University, New York 27, New York

Graham, Evarts Ambrose, 1941 (10), Barnes Hospital, 600 South Kingshighway, St. Louis 10, Missouri

Greenewalt, Crawford Hallock, 1952 (4), E. I. du Pont de Nemours and Company, Incorporated, Wilmington 98, Delaware

Gregory, William King, 1927 (6), American Museum of Natural History, 77th Street and Central Park West, New York 24, New York

Griggs, David Tressel, 1952 (13), Institute of Geophysics, University of California, Los Angeles 24, California

Guilford, Joy Paul, 1954 (12), P. O. Box 1288, Beverly Hills, California

Gunn, Ross, 1951 (13), United States Weather Bureau, Department of Commerce, Washington 25, D. C.

Gutenberg, Beno, 1945 (13), Seismological Laboratory, 220 North San Rafael Avenue, Pasadena 2, California

Hamburger, Viktor, 1953 (8), Department of Zoology, Washington University, St. Louis 5, Missouri

Hammett, Louis Plack, 1943 (5), Department of Chemistry, Columbia University, New York 27, New York

Harlow, Harry F., 1951 (12), Department of Psychology, University of Wisconsin, Madison 6, Wisconsin

Harned, Herbert Spencer, 1950 (5), Sterling Chemistry Laboratory, Yale University, 225 Prospect Street, New Haven 11, Connecticut

Harrison, Ross Granville, 1913 (8), Osborn Zoological Laboratory, Yale University, New Haven 11, Connecticut

Hartline, Haldan Keffer, 1948 (9), Rockefeller Institute for Medical Research, New York 21, New York

Hartman, Carl Gottfried, 1937 (8), Ortho Research Foundation, Raritan, New Jersey

Harvey, Edmund Newton, 1934 (8), Guyot Hall, Princeton University, Princeton, New Jersey

Hastings, Albert Baird, 1939 (9), Department of Biological Chemistry, Harvard Medical School, 25 Shattuck Street, Boston 15, Massachusetts

Heidelberger, Michael, 1942 (10), Institute of Microbiology, Rutgers University, New Brunswick, New Jersey

Hendricks, Sterling Brown, 1952 (5), Plant Industry Station, Beltsville, Maryland

Herb, Raymond George, 1955 (3), Department of Physics, University of Wisconsin, Madison 6, Wisconsin

Herrick, Charles Judson, 1918 (8), 236 Morningside Drive, S. E., Grand Rapids 6, Michigan

Hess, Harry Hammond, 1952 (6), Department of Geology, Princeton University, Princeton, New Jersey

Hewett, Donnel Foster, 1937 (6), 1460 Rose Villa Street, Pasadena 4, California

Hildebrand, Joel Henry, 1929 (5), Department of Chemistry, University of California, Berkeley 4, California

Hilgard, Ernest Ropiequet, 1948 (12), Department of Psychology, Stanford University, Stanford, California

Hille, Carl Einar, 1953 (1), Department of Mathematics, Yale University, New Haven 11, Connecticut

Hirschfelder, Joseph Oakland, 1953 (5), University of Wisconsin, P. O. Box 2127, Madison 6, Wisconsin

Hisaw, Frederick Lee, 1947 (8), The Biological Laboratories, Harvard University, 16 Divinity Avenue, Cambridge 38, Massachusetts

Holtfreter, Johannes, 1955 (8), Biological Laboratories, University of Rochester, Rochester 3, New York

Hoover, Herbert Clark, 1922 (4), Waldorf Astoria Towers, New York, New York

Horsfall, Frank Lappin, Jr., 1948 (10), Rockefeller Institute for Medical Research, New York 21, New York

Horsfall, James Gordon, 1953 (7), Connecticut Agricultural Experiment Station, New Haven 4, Connecticut

Houston, William Vermillion, 1943 (3), The Rice Institute, Houston, Texas

Hubbert, Marion King, 1955 (6), Shell Oil Company, P. O. Box 2099, Houston 1, Texas

Hubbs, Carl Leavitt, 1952 (8), Scripps Institution of Oceanography, La Jolla, California

Huggins, Charles Brenton, 1949 (10), Department of Surgery, University of Chicago, 950 East 59th Street, Chicago 37, Illinois

Hull, Albert Wallace, 1929 (3), General Electric Research Laboratory, The Knolls, Schenectady, New York

Hunsaker, Jerome Clark, 1935 (4), Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Hutchinson, George Evelyn, 1950 (8), Osborn Zoological Laboratory, Yale University, New Haven 11, Connecticut

Irwin, Malcolm Robert, 1950 (8), Department of Genetics, University of Wisconsin, Madison 6, Wisconsin

Iselin, Columbus O'Donnell, 1951 (13), Woods Hole Oceanographic Institution, Woods Hole, Massachusetts

Jacobs, Merkel Henry, 1939 (8), School of Medicine, University of Pennsylvania, Philadelphia 4, Pennsylvania

Jacobs, Walter Abraham, 1932 (5), Rockefeller Institute for Medical Research, New York 21, New York

Jacobson, Nathan, 1954 (1), Department of Mathematics, Yale University, New Haven 11, Connecticut

Jeffries, Zay, 1939 (4), General Electric Company, 1 Plastics Avenue, Pittsfield, Massachusetts

Johnson, John Raven, 1948 (5), Department of Chemistry, Cornell University, Ithaca, New York

Johnson, William Summer, 1952 (5), Department of Chemistry, University of Wisconsin, Madison 6, Wisconsin

Jones, Donald **F**orsha, 1939 (7), Department of Genetics, Connecticut Agricultural Experiment Station, P. O. Box 1106, New Haven 4, Connecticut

Joy, Alfred Harrison, 1944 (2), Mount Wilson Observatory, Pasadena 4, California

Kaufmann, Berwind Petersen, 1952 (7), Department of Genetics, Carnegie Institution of Washington, Cold Spring Harbor, Long Island, New York

Kelley, Walter **P**earson, 1943 (6), 120 Hilgard Hall, University of California, Berkeley 4, California

Kellogg, Arthur **Remington,** 1951 (8), United States National Museum, Smithsonian Institution, Washington 25, D. C.

Kelly, Mervin J., 1945 (4), Bell Telephone Laboratories, Incorporated, 463 West Street, New York 14, New York

Kemble, Edwin Crawford, 1931 (3), Physics Laboratories, Harvard University, Cambridge 38, Massachusetts

Kendall, Edward Calvin, 1950 (14), 3 Queenston Place, Princeton, New Jersey

Kent, Robert **H**arrington, 1951 (3), Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland

Kerst, Donald William, 1951 (3), Department of Physics, University of Illinois, Urbana, Illinois

Kettering, Charles **F**ranklin, 1928 (4), General Motors Corporation Research Laboratories, Box 188, North End Station, Detroit 2, Michigan

Keyes, Frederick **G**eorge, 1930 (5), Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Kharasch, Morris **S**elig, 1946 (5), Department of Chemistry, University of Chicago, Chicago 37, Illinois

Kidder, Alfred **V**incent, 1936 (11), 41 Holden Street, Cambridge 38, Massachusetts

Kimball, George Elbert, 1954 (5), Department of Chemistry, Columbia University, New York 27, New York

King, Arthur Scott, 1941 (2), Mount Wilson Observatory, Pasadena 4, California

King, Charles **G**len, 1951 (14), The Nutrition Foundation, Incorporated, 99 Park Avenue, New York 16, New York

Kirkwood, John Gamble, 1942 (5), Sterling Chemistry Laboratory, Yale University, 225 Prospect Street, New Haven 11, Connecticut

Kistiakowsky, George **B**ogdan, 1939 (5), Department of Chemistry, Harvard University, 12 Oxford Street, Cambridge 38, Massachusetts

Kluckhohn, Clyde Kay Maben, 1952 (11), Peabody Museum, Cambridge 38, Massachusetts

Knopf, Adolph, 1931 (6), Department of Geology, Stanford University, Stanford, California

Köhler, Wolfgang, 1947 (12), The Institute for Advanced Study, Princeton, New Jersey

Kraus, Charles August, 1925 (5), Brown University, Providence 12, Rhode Island

Kroeber, Alfred L., 1928 (11), Department of Anthropology, University of California, Berkeley 4, California

Kuiper, Gerard Peter, 1950 (2), Yerkes Observatory, University of Chicago, Williams Bay, Wisconsin

Kunkel, Louis Otto, 1932 (7), Rockefeller Institute for Medical Research, New York 21, New York

Lamb, Willis Eugene, Jr., 1954 (3), Department of Physics, Stanford University, Stanford, California

Lambert, Walter Davis, 1949 (13), P. O. Box 687, Canaan, Connecticut

LaMer, Victor Kuhn, 1945 (5), Department of Chemistry, Columbia University, New York 27, New York

Land, Edwin Herbert, 1953 (3), Polaroid Corporation, Cambridge 39, Massachusetts

Landis, Eugene Markley, 1954 (9), Department of Physiology, Harvard Medical School, 25 Shattuck Street, Boston 15, Massachusetts

Langmuir, Irving, 1918 (5), Research Laboratory, General Electric Company, P. O. Box 1088, Schenectady, New York

Larsen, Esper S., Jr., 1944 (6), Apt. 502 H, 3930 Connecticut Avenue, N. W., Washington 8, D. C.

Lashley, Karl Spencer, 1930 (12), Yerkes Laboratories of Primate Biology, Incorporated, Orange Park, Florida

Lauritsen, Charles Christian, 1941 (3), California Institute of Technology, Pasadena 4, California

Lawrence, Ernest Orlando, 1934 (3), Radiation Laboratory, University of California, Berkeley 4, California

Lefschetz, Solomon, 1925 (1), Fine Hall, Princeton University, Princeton, New Jersey

Leith, Charles Kenneth, 1920 (6), Sheraton Park Hotel, Washington 8, D. C.

Leonard, Nelson Jordan, 1955 (5), Department of Chemistry and Chemical Engineering, University of Illinois, Urbana, Illinois

Lewis, Warren Harmon, 1936 (8), The Wistar Institute of Anatomy and Biology, Philadelphia 4, Pennsylvania

Lewis, Warren Kendall, 1938 (4), Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Libby, Willard Frank, 1950 (5), 3101 Hawthorne Street, N. W., Washington 8, D. C.

Lind, Samuel Colville, 1930 (5), P. O. Box P, Oak Ridge, Tennessee

Lindsley, Donald Benjamin, 1952 (12), Department of Psychology, University of California, Los Angeles 24, California

Link, Karl Paul, 1946 (14), Department of Biochemistry, University of Wisconsin, Madison 6, Wisconsin

Lipmann, Fritz Albert, 1950 (14), Biochemical Research Laboratory, Massachusetts General Hospital, Boston 14, Massachusetts

Little, Clarence **C**ook, 1945 (10), Roscoe B. Jackson Memorial Laboratory, Bar Harbor, Maine

Lloyd, David Pierce **C**aradoc, 1953 (9), Rockefeller Institute for Medical Research, New York 21, New York

Loeb, Leo, 1937 (10), 40 Crestwood Drive, St. Louis 5, Missouri

Loeb, Robert Frederick, 1946 (9), Columbia University College of Physicians and Surgeons, 620 West 168th Street, New York 32, New York

Long, Cyril **N**orman **H**ugh, 1948 (9), Yale University School of Medicine, 333 Cedar Street, New Haven 11, Connecticut

Long, Esmond Ray, 1946 (10), Henry Phipps Institute, 7th and Lombard Streets, Philadelphia 47, Pennsylvania

Longsworth, Lewis Gibson, 1947 (5), Rockefeller Institute for Medical Research, New York 21, New York

Longwell, Chester Ray, 1935 (6), United States Geological Survey, Menlo Park, California

Loomis, Alfred Lee, 1941 (4), The Loomis Institute for Scientific Research, Incorporated, Room 2420, 14 Wall Street, New York 5, New York

Loomis, Francis **W**heeler, 1949 (3), 804 West Illinois Street, Urbana, Illinois

Lorente de Nó, Rafael, 1950 (9), Rockefeller Institute for Medical Research, New York 21, New York

Lothrop, Samuel **K**irkland, 1951 (11), Peabody Museum, Harvard University, Cambridge 38, Massachusetts

Lovering, Thomas Seward, 1949 (6), 8001 West 17th Avenue, Lakewood, Colorado

Lowie, Robert Harry, 1931 (11), Department of Anthropology, University of California, Berkeley 4, California

Macelwane, James Bernard, **S.J.,** 1944 (13), 3621 Olive Street, St. Louis 8, Missouri

MacInnes, Duncan Arthur, 1937 (5), Rockefeller Institute for Medical Research, New York 21, New York

Mac Lane, Saunders, 1949 (1), Department of Mathematics, University of Chicago, Chicago 37, Illinois

MacLeod, Colin Munro, 1955 (10), Department of Microbiology, New York University College of Medicine, 550 First Avenue, New York 16, New York

Magoun, Horace **W**inchell, 1955 (9), Department of Anatomy, University of California Medical Center, Los Angeles 24, California

Mangelsdorf, Paul **C**hristoph, 1945 (7), Botanical Museum, Harvard University, Cambridge 38, Massachusetts

Mann, Frank Charles, 1950 (9), 605 11th Street, S. W., Rochester, Minnesota

Marshall, Eli **K**ennerly, **Jr.,** 1943 (9), Department of Medicine, Johns Hopkins Hospital, Baltimore 5, Maryland

Marvel, Carl **S**hipp, 1938 (5), Department of Chemistry, University of Illinois, Urbana, Illinois

Mason, Max, 1923 (3), 1035 Harvard Avenue, Claremont, California

Maxcy, Kenneth Fuller, 1950 (10), 700 St. Georges Road, Baltimore 10, Maryland

Mayall, Nicholas **U**lrich, 1949 (2), Lick Observatory, Mount Hamilton, California

Mayer, Joseph Edward, 1946 (5), Institute for Nuclear Studies, University of Chicago, Chicago 37, Illinois

Maynard, Leonard Amby, 1944 (14), Savage Hall, Cornell University, Ithaca, New York

Mayr, Ernst, 1954 (8), Museum of Comparative Zoology at Harvard College, Oxford Street, Cambridge 38, Massachusetts

McClintock, Barbara, 1944 (7), Carnegie Institution of Washington, Cold Spring Harbor, Long Island, New York

McCollum, Elmer Verner, 1920 (14), Gilman Hall, Johns Hopkins University, Baltimore 18, Maryland

McElvain, Samuel Marion, 1949 (5), Department of Chemistry, University of Wisconsin, Madison 6, Wisconsin

McMaster, Philip Duryeé, 1952 (10), Rockefeller Institute for Medical Research, New York 21, New York

McMillan, Edwin Mattison, 1947 (3), Radiation Laboratory, University of California, Berkeley 4, California

McShane, Edward James, 1948 (1), Department of Mathematics, University of Virginia, Charlottesville, Virginia

Mead, Warren Judson, 1939 (6), 88 Rutledge Road, Belmont 78, Massachusetts

Meek, Walter Joseph, 1947 (9), Department of Physiology, University of Wisconsin, Madison 6, Wisconsin

Mees, Charles Edward Kenneth, 1950 (5), Eastman Kodak Company, Kodak Park Works, Rochester 4, New York

Meggers, William Frederick, 1954 (3), National Bureau of Standards, Washington 25, D. C.

Mendenhall, Walter Curran, 1932 (6), 9 East Lenox Street, Chevy Chase 15, Maryland

Menzel, Donald Howard, 1948 (2), Harvard College Observatory, Cambridge 38, Massachusetts

Merica, Paul Dyer, 1942 (4), 67 Wall Street, New York 5, New York

Merrill, Paul Willard, 1929 (2), 813 Santa Barbara Street, Pasadena 4, California

Metz, Charles William, 1948 (8), Department of Zoology, University of Pennsylvania, Philadelphia 4, Pennsylvania

Meyer, Karl Friederich, 1940 (10), George Williams Hooper Foundation, University of California Medical Center, San Francisco 22, California

Miles, Walter Richard, 1933 (12), Faculty of Letters, Istanbul University, Istanbul, Turkey

Mirsky, Alfred Ezra, 1954 (8), Rockefeller Institute for Medical Research, New York 21, New York

Mitchell, Samuel Alfred, 1933 (2), Leander McCormick Observatory, University Station, Charlottesville, Virginia

Montgomery, Deane, 1955 (1), School of Mathematics, The Institute for Advanced Study, Princeton, New Jersey

Moore, Carl Richard, 1944 (8), University of Chicago, Chicago 37, Illinois

Moore, Robert Lee, 1931 (1), University of Texas, Austin 12, Texas

Morse, Harold Marston, 1932 (1), The Institute for Advanced Study, Princeton, New Jersey

Morse, Philip McCord, 1955 (3), Department of Physics, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Muller, Hermann Joseph, 1931 (8), Zoology Department, Indiana University, Bloomington, Indiana.

Mulliken, Robert Sanderson, 1936 (3), Laboratory of Molecular Structure and Spectra, Department of Physics, University of Chicago, Chicago 37, Illinois

Murnaghan, Francis Dominic, 1942 (1), 6202 Sycamore Road, Baltimore 12, Maryland

Murphree, Eger Vaughan, 1950 (4), Esso Research and Engineering Company, 15 West 51st Street, New York 19, New York

Nicholas, John Spangler, 1949 (8), Osborn Zoological Laboratory, Yale University, New Haven 11, Connecticut

Nicholson, Seth Barnes, 1937 (2), Mount Wilson Observatory, Pasadena 4, California

Niemann, Carl George, 1952 (5), Gates and Crellin Laboratories of Chemistry, California Institute of Technology, Pasadena 4, California

Nier, Alfred Otto C., 1950 (3), Department of Physics, University of Minnesota, Minneapolis 14, Minnesota

Nissen, Henry W., 1953 (12), Yerkes Laboratories of Primate Biology, Incorporated, Orange Park, Florida

Nolan, Thomas Brennan, 1951 (6), United States Geological Survey, Department of the Interior, Washington 25, D. C.

Northrop, John Howard, 1934 (14), Department of Bacteriology, University of California, Berkeley 4, California

Noyes, William Albert, Jr., 1943 (5), Department of Chemistry, University of Rochester, Rochester 3, New York

O'Brien, Brian, 1954 (3), American Optical Company, Southbridge, Massachusetts

Oncley, John Lawrence, 1947 (14), Department of Biological Chemistry, Harvard Medical School, 25 Shattuck Street, Boston 15, Massachusetts

Onsager, Lars, 1947 (5), Sterling Chemistry Laboratory, Yale University, New Haven, Connecticut

Opie, Eugene Lindsay, 1923 (10), Rockefeller Institute for Medical Research, New York 21, New York

Oppenheimer, J. Robert, 1941 (3), The Institute for Advanced Study, Princeton, New Jersey

Osterhout, Winthrop John Vanleuven, 1919 (7), Rockefeller Institute for Medical Research, New York 21, New York

Painter, Theophilus Shickel, 1938 (8), University of Texas, Austin 12, Texas

Panofsky, Wolfgang K. H., 1954 (3), High-Energy Physics Laboratory, Stanford University, Stanford, California

Patterson, John Thomas, 1941 (8), Department of Zoology, University of Texas, Austin 12, Texas

Paul, John Rodman, 1945 (10), Yale University School of Medicine, 333 Cedar Street, New Haven 11, Connecticut

Pauling, Linus, 1933 (5), Gates and Crellin Laboratories of Chemistry, California Institute of Technology, Pasadena 4, California

Pegram, George Braxton, 1949 (3), Columbia University, New York 27, New York

Pekeris, Chaim Leib, 1952 (13), Department of Applied Mathematics, The Weizmann Institute of Science, Rehovot, Israel

Peters, John Punnett, 1947 (9), Department of Internal Medicine, Yale University School of Medicine, 333 Cedar Street, New Haven 11, Connecticut

Petrunkevitch, Alexander, 1954 (8), Osborn Zoological Laboratory, Yale University, New Haven 11, Connecticut

Pierce, George Washington, 1920 (3), 7 Berkeley Place, Cambridge 38, Massachusetts

Pierce, John Robinson, 1955 (4), Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey

Piggot, Charles Snowden, 1946 (13), 2200 Decatur Place, N. W., Washington 8, D. C.

Pillsbury, Walter Bowers, 1925 (12), Haven Hall, University of Michigan, Ann Arbor, Michigan

Pitzer, Kenneth Sanborn, 1949 (5), Department of Chemistry and Chemical Engineering, University of California, Berkeley 4, California

Purcell, Edward Mills, 1951 (3), Lyman Laboratory of Physics, Harvard University, Cambridge 38, Massachusetts.

Rabi, Isidor Isaac, 1940 (3), Department of Physics, Columbia University, New York 27, New York

Ramsey, Norman Foster, 1952 (3), Lyman Laboratory of Physics, Harvard University, Cambridge 38, Massachusetts

Raper, Kenneth Bryan, 1949 (7), Department of Bacteriology, University of Wisconsin, Madison 6, Wisconsin

Raymond, Arthur Emmons, 1950 (4), Douglas Aircraft Company, Incorporated, 3000 Ocean Park Boulevard, Santa Monica, California

Reeside, John Bernard, Jr., 1945 (6), Box 177, Hyattsville, Maryland

Reichelderfer, Francis Wilton, 1945 (13), United States Weather Bureau, Department of Commerce, Washington 25, D. C.

Rhoades, Marcus Morton, 1946 (7), Department of Botany, University of Illinois, Urbana, Illinois

Rich, Arnold Rice, 1954 (10), Johns Hopkins Hospital, Baltimore 5, Maryland

Richards, Alfred Newton, 1927 (9), School of Medicine, University of Pennsylvania, Philadelphia 4, Pennsylvania

Richter, Curt Paul, 1948 (12), Johns Hopkins Hospital, Baltimore 5, Maryland

Riddle, Oscar, 1939 (8), Route 4, Plant City, Florida

Riker, Albert Joyce, 1951 (7), Department of Plant Pathology, University of Wisconsin, Madison 6, Wisconsin

Rittenberg, David, 1953 (14), College of Physicians and Surgeons, 630 West 168th Street, New York 32, New York

Rivers, Thomas Milton, 1934 (10), Rockefeller Institute for Medical Research, New York 21, New York

Robbins, William Jacob, 1940 (7), New York Botanical Garden, Bronx Park (Fordham Station), New York 58, New York

Robertson, Howard Percy, 1951 (3), OSCAD, SHAPE, A. P. O. 55, New York, New York

Robertson, Oswald Hope, 1943 (10), Department of Biology, Stanford University, Stanford, California

Rodebush, Worth Huff, 1938 (5), University of Illinois, Urbana, Illinois

Romer, Alfred Sherwood, 1944 (8), Museum of Comparative Zoology at Harvard College, Oxford Street, Cambridge 38, Massachusetts

Rose, William Cumming, 1936 (14), University of Illinois, Urbana, Illinois

Ross, Frank Elmore, 1930 (2), 1100 Mt. Lowe Drive, Altadena, California

Rossby, Carl-Gustaf Arvid, 1943 (13), Institute of Meteorology, University of Stockholm, Stockholm, Sweden

Rossi, Bruno Benedetto, 1950 (3), Department of Physics, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Rossini, Frederick Dominic, 1951 (5), Carnegie Institute of Technology, Pittsburgh 13, Pennsylvania

Rous, Francis **Peyton,** 1927 (10), Rockefeller Institute for Medical Research, New York 21, New York

Rubey, William Walden, 1945 (6), United States Geological Survey, Department of the Interior, Washington 25, D. C.

Ruedemann, Rudolf, 1928 (6), 161 Dana Avenue, Albany 3, New York

Russell, Henry Norris, 1918 (2), Princeton University, Princeton, New Jersey

Sabin, Albert Bruce, 1951 (10), The Children's Hospital Research Foundation, Elland Avenue and Bethesda, Cincinnati 29, Ohio

Savage, John Lucian, 1949 (4), 1651 Dahlia Street, Denver 7, Colorado

Sax, Karl, 1941 (7), Arnold Arboretum, Jamaica Plain 30, Massachusetts

Scatchard, George, 1946 (5), Department of Chemistry, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Schairer, John Frank, 1953 (6), Geophysical Laboratory, Carnegie Institution of Washington, 2801 Upton Street, N. W., Washington 8, D. C.

Schlesinger, Hermann Irving, 1948 (5), Department of Chemistry, University of Chicago, Chicago 37, Illinois

Schmidt, Carl Frederic, 1949 (9), Laboratory of Pharmacology, School of Medicine, University of Pennsylvania, Philadelphia 4, Pennsylvania

Schmitt, Francis Otto, 1948 (8), Department of Biology, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Schrader, Franz, 1951 (8), Department of Zoology, Columbia University, New York 27, New York

Schwinger, Julian, 1949 (3), Department of Physics, Harvard University, Cambridge 38, Massachusetts

Seaborg, Glenn Theodore, 1948 (5), Department of Chemistry, University of California, Berkeley 4, California

Seares, Frederick Hanley, 1919 (2), 351 Palmetto Drive, Pasadena 2, California

Segrè, Emilio, 1952 (3), Department of Physics, University of California, Berkeley 4, California

Seitz, Frederick, 1951 (3), Department of Physics, University of Illinois, Urbana. Illinois

Serber, Robert, 1952 (3), Department of Physics, Columbia University, New York 27, New York

Shaffer, Philip Anderson, 1928 (14), School of Medicine, Washington University, Euclid Avenue and Kingshighway, St. Louis 10, Missouri

Shapiro, Harry Lionel, 1949 (11), American Museum of Natural History, Central Park West at 79th Street, New York 24, New York

Shapley, Harlow, 1924 (2), Sharon Cross Road, Peterboro, New Hampshire

Shedlovsky, Theodore, 1953 (5), Rockefeller Institute for Medical Research, New York 21, New York

Sherman, Henry Clapp, 1933 (14), 1021 Park Avenue, New York 28, New York

Shockley, William, 1951 (4), Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey

Shope, Richard Edwin, 1940 (10), Rockefeller Institute for Medical Research, New York 21, New York

Simpson, George Gaylord, 1941 (6), American Museum of Natural History, 77th Street and Central Park West, New York 24, New York

Sinnott, Edmund Ware, 1936 (7), Hall of Graduate Studies, Yale University, New Haven 11, Connecticut

Skinner, Burrhus Frederic, 1950 (12), Memorial Hall, Harvard University, Cambridge 38, Massachusetts

Slater, John Clarke, 1932 (3), Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Slepian, Joseph, 1941 (4), Westinghouse Research Laboratories, East Pittsburgh, Pennsylvania

Slichter, Louis Byrne, 1944 (13), Institute of Geophysics, University of California, Los Angeles 24, California

Slipher, Vesto Melvin, 1921 (2), Lowell Observatory, Flagstaff, Arizona

Small, Lyndon Frederick, 1941 (5), National Institutes of Health, Bethesda 14, Maryland

Smith, Gilbert Morgan, 1948 (7), Department of Biology, Stanford University, Stanford, California

Smith, Homer William, 1945 (9), 550 First Avenue, New York 16, New York

Smith, Lee Irvin, 1944 (5), School of Chemistry, University of Minnesota, Minneapolis 14, Minnesota

Smith, Paul Althaus, 1947 (1), Department of Mathematics, Columbia University, New York 27, New York

Smith, Phillip Edward, 1939 (8), R. D., Sunderland, Massachusetts

Smyth, Charles Phelps, 1955 (5), Department of Chemistry, Princeton University, Princeton, New Jersey

Snell, Esmond Emerson, 1955 (14), Department of Chemistry, University of Texas, Austin 12, Texas

Soderberg, Carl Richard, 1947 (4), Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Sonneborn, Tracy Morton, 1946 (8), Jordan Hall, Indiana University, Bloomington, Indiana

Spedding, Frank Harold, 1952 (5), Box 14-A, Station A, Institute for Atomic Research, Iowa State College, Ames Iowa

Spence, Kenneth Wartinbe, 1955 (12), Department of Psychology, State University of Iowa, Iowa City, Iowa

Spier, Leslie, 1946 (11), P. O. Box 880, Santa Cruz, California

Spitzer, Lyman, Jr., 1952 (2), Princeton University Observatory, 14 Prospect Avenue, Princeton, New Jersey

Stadie, William Christopher, 1945 (14), 820 Maloney Clinic Building, 36th and Spruce Streets, Philadelpha 4, Pennsylvania

Stakman, Elvin Charles, 1934 (7), Institute of Agriculture, University of Minnesota, St. Paul 1, Minnesota

Stanley, Wendell Meredith, 1941 (14), Biochemistry and Virus Laboratory, University of California, Berkeley 4, California

Stebbins, George **Ledyard,** 1952 (7), Department of Genetics, University of California, Davis, California

Stebbins, Joel, 1920 (2), Lick Observatory, Mt. Hamilton, California

Stern, Curt, 1948 (8), Department of Zoology, University of California, Berkeley 4, California

Stern, Otto, 1945 (3), 759 Cragmont Avenue, Berkeley 8, California

Stevens, Stanley Smith, 1946 (12), Memorial Hall, Harvard University, Cambridge 38, Massachusetts

Steward, Julian H., 1954 (11), Department of Sociology and Anthropology, University of Illinois, Urbana, Illinois

Stewart, George Walter, 1938 (3), Hall of Physics, State University of Iowa, Iowa City, Iowa

Stone, Marshall Harvey, 1938 (1), 303 Eckhart Hall, University of Chicago, Chicago 37, Illinois

Stratton, Julius Adams, 1950 (4), Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Street, Jabez Curry, 1953 (3), Jefferson Physical Laboratory, Harvard University, Cambridge 38, Massachusetts

Struve, Otto, 1937 (2), Department of Astronomy, University of California, Berkeley 4, California

Sturtevant, Alfred Henry, 1930 (8), California Institute of Technology, Pasadena 4, California

Suits, Chauncey Guy, 1946 (4), General Electric Research Laboratory, The Knolls, Schenectady, New York

Sumner, James Batcheller, 1948 (14), 508 Cayuga Heights Road, Ithaca, New York

Taliaferro, William Hay, 1940 (8), University of Chicago, 5724 Ellis Avenue, Chicago 37, Illinois

Tatum, Edward Lawrie, 1952 (14), Department of Biological Sciences, Stanford University, Stanford, California

Teller, Edward, 1948 (3), Department of Physics, University of California, Berkeley 4, California

Terman, Frederick Emmons, 1946 (4), School of Engineering, Stanford University, Stanford, California

Terman, Lewis Madison, 1928 (12), 761 Dolores Street, Stanford University, Stanford, California

Thimann, Kenneth Vivian, 1948 (7), The Biological Laboratories, Harvard University, 16 Divinity Avenue, Cambridge 38, Massachusetts

Thom, Charles, 1937 (7), Port Jefferson, New York

Thomas, Charles Allen, 1948 (4), Monsanto Chemical Company, 1700 South Second Street, St. Louis 4, Missouri

Thomas, Tracy Yerkes, 1941 (1), Swain Hall, Indiana University, Bloomington, Indiana

Thompson, Thomas Gordon, 1951 (13), Department of Oceanography, University of Washington, Seattle 5, Washington

Thurstone, Louis Leon, 1938 (12), Psychometric Laboratory, University of North Carolina, Chapel Hill, North Carolina

Tillett, William Smith, 1951 (10), New York University College of Medicine, 550 First Avenue, New York 16, New York

Timoshenko, Stephen Prokop, 1940 (4), Room 262, Engineering Building, Stanford University, Stanford, California

Tishler, Max, 1953 (5), Chemical Division, Merck & Co., Inc., Rahway, New Jersey

Tolman, Edward Chace, 1937 (12), Department of Psychology, University of California, Berkeley 4, California

Trumpler, Robert Julius, 1932 (2), Route 2, Box 1750, Aptos, California

Tuve, Merle Antony, 1946 (3), Department of Terrestrial Magnetism, Carnegie Institution of Washington, 5241 Broad Branch Road, N. W., Washington 15, D. C.

Twitty, Victor Chandler, 1950 (8), Department of Biological Sciences, Stanford University, Stanford, California

Uhlenbeck, George Eugene, 1955 (3), Department of Physics, University of Michigan, Ann Arbor, Michigan

Urey, Harold Clayton, 1935 (5), Institute for Nuclear Studies, University of Chicago, Chicago 37, Illinois

Van Niel, Cornelis Bernardus, 1945 (7), Hopkins Marine Station of Stanford University, Pacific Grove, California

Van Slyke, Donald Dexter, 1921 (14), Brookhaven National Laboratory, Upton, Long Island, New York

Van Vleck, John Hasbrouck, 1935 (3), Lyman Laboratory of Physics, Harvard University, Cambridge 38, Massachusetts

Veblen, Oswald, 1919 (1), The Institute for Advanced Study, Princeton, New Jersey

Vestine, Ernest Harry, 1954 (13), Department of Terrestrial Magnetism, Carnegie Institution of Washington, 5241 Broad Branch Road, N. W., Washington 15, D. C.

Vickery, Hubert Bradford, 1943 (14), Connecticut Agricultural Experiment Station, New Haven 4, Connecticut

von Kármán, Theodore, 1938 (4), 1501 South Marengo Avenue, Pasadena, California

von Neumann, John, 1937 (1), United States Atomic Energy Commission, Room 207, Washington 25, D. C.

Waksman, Selman Abraham, 1942 (7), Institute of Microbiology, Rutgers University, New Brunswick, New Jersey

Wald, George, 1950 (14), The Biological Laboratories, Harvard University, 16 Divinity Avenue, Cambridge 38, Massachusetts

Walker, John Charles, 1945 (7), 206 Horticulture Building, University of Wisconsin, Madison 6, Wisconsin

Walsh, Joseph Leonard, 1936 (1), Widener 474, Harvard University, Cambridge 38, Massachusetts

Webster, David Locke, 1923 (3), 1830 Cowper Street, Palo Alto., California

Weiss, Paul Alfred, 1947 (8), Rockefeller Institute for Medical Research, New York 21, New York

Weisskopf, Victor Frederick, 1952 (3), Department of Physics, Massachusetts Institute of Technology, Cambridge 39, Massachusetts

Went, Frits Warmolt, 1947 (7), California Institute of Technology, Pasadena 4, California

Werkman, Chester Hamlin, 1946 (14), Department of Bacteriology, Iowa State College, Ames, Iowa

Westheimer, Frank Henry, 1954 (5), Department of Chemistry, Harvard University, 12 Oxford Street, Cambridge 38, Massachusetts

Wetmore, Alexander, 1945 (8), Smithsonian Institution, Washington 25, D. C.

Wetmore, Ralph Hartley, 1954 (7), The Biological Laboratories, Harvard University, 16 Divinity Avenue, Cambridge 38, Massachusetts

Wever, Ernest Glen, 1940 (12), Princeton University, Princeton, New Jersey

Weyl, Claus Hugo **Hermann,** 1940 (1), The Institute for Advanced Study, Princeton, New Jersey

Wheeler, John Archibald, 1952 (3), Palmer Physical Laboratory, Princeton University, Princeton, New Jersey

Whipple, George Hoyt, 1929 (10), School of Medicine and Dentistry, University of Rochester, 260 Crittenden Boulevard, Rochester 20, New York

Whitford, Albert Edward, 1954 (2), Washburn Observatory, University of Wisconsin, Madison 6, Wisconsin

Whitney, Hassler, 1945 (1), The Institute for Advanced Study, Princeton, New Jersey

Whyburn, Gordon Thomas, 1951 (1), Department of Mathematics, University of Virginia, Charlottesville, Virginia

Wiggers, Carl John, 1951 (9), Cleveland Clinic, 2020 East 93rd Street, Cleveland 6, Ohio

Wigner, Eugene Paul, 1945 (3), 8 Ober Road, Princeton, New Jersey

Williams, Howel, 1950 (6), Department of Geological Sciences, University of California, Berkeley 4, California

Williams, John Warren, 1952 (5), Department of Chemistry, University of Wisconsin, Madison 6, Wisconsin

Williams, Robert R., 1945 (5), 297 Summit Avenue, Summit, New Jersey

Williams, Robley Cook, 1955 (14), Virus Laboratory, University of California, Berkeley 4, California

Williams, Roger John, 1946 (5), Biochemical Institute, University of Texas, Austin 12, Texas

Willier, Benjamin Harrison, 1945 (8), Department of Biology, Johns Hopkins University, Baltimore 18, Maryland

Wilson, David Wright, 1955 (14), Department of Biochemistry, School of Medicine, University of Pennsylvania, Philadelphia 4, Pennsylvania

Wilson, Edgar Bright, Jr., 1947 (5), Department of Chemistry, Harvard University, 12 Oxford Street, Cambridge 38, Massachusetts

Wilson, Edwin Bidwell, 1919 (3), Harvard School of Public Health, 695 Huntington Avenue, Boston 15, Massachusetts

Wilson, Perry William, 1955 (7), Department of Bacteriology, University of Wisconsin, Madison 6, Wisconsin

Wilson, Ralph Elmer, 1950 (2), 605 Acacia Avenue, Corona del Mar, California

Wilson, Robert Erastus, 1947 (4), 910 South Michigan Avenue, Chicago 80, Illinois

Winstein, Saul, 1955 (5), Department of Chemistry, University of California, Los Angeles 24, California

Wintersteiner, Oskar, 1950 (14), Squibb Institute for Medical Research, New Brunswick, New Jersey

Wislocki, George Bernays, 1941 (8), Harvard Medical School, 25 Shattuck Street, Boston 15, Massachusetts

Wolfrom, Melville, Lawrence, 1950 (5), Department of Chemistry, The Ohio State University, Columbus 10, Ohio

Wood, Harland Goff, 1953 (14), Department of Special Pathology and Therapy, The Royal Veterinary and Agricultural College, Copenhagen, Denmark. After February 1, 1956: Department of Biochemistry, School of Medicine, Western Reserve University, Cleveland 6, Ohio

Wood, Robert Williams, 1912 (3), Johns Hopkins University, Baltimore 18, Maryland

Woodring, Wendell Phillips, 1946 (6), United States Geological Survey, Department of the Interior, Washington 25, D. C.

Woodward, Robert Burns, 1953 (5), Department of Chemistry, Harvard University, 12 Oxford Street, Cambridge 38, Massachusetts

Woodworth, Robert Sessions, 1921 (12), 400 West 119th Street, New York 27, New York

Woolley, Dilworth Wayne, 1952 (14), Rockefeller Institute for Medical Research, New York 21, New York

Wright, Sewall Green, 1934 (8), Department of Genetics, University of Wisconsin, Madison 6, Wisconsin

Wright, William Hammond, 1922 (2), 60 North Keeble Avenue, San Jose 26, California

Wulf, Oliver Reynolds, 1949 (13), United States Weather Bureau, Gates and Crellin Laboratories of Chemistry, California Institute of Technology, Pasadena 4, California

Wyckoff, Ralph Walter Graystone, 1949 (5), Laboratory of Physical Biology, National Institutes of Health, Bethesda 14, Maryland

Yerkes, Robert Mearns, 1923 (12), 4 St. Ronan Terrace, New Haven 11, Connecticut

Yost, Don Merlin Lee, 1944 (5), California Institute of Technology, Pasadena 4, California

Young, William Gould, 1951 (5), Department of Chemistry, University of California, Los Angeles 24, California

Zachariasen, Frederik **William** Houlder, 1949 (3), Department of Physics, University of Chicago, Chicago 37, Illinois

Zariski, Oscar, 1944 (1), Department of Mathematics, Harvard University, Cambridge 38, Massachusetts

Zworykin, Vladimir **K**osma, 1943 (4), RCA Laboratories Division, David Sarnoff Research Center, Princeton, New Jersey

Number of Members July 1, 1955: 542.

MEMBERS EMERITI

Anderson, John **A**ugust, 1928 (2), P. O. Box 332, Pasadena 17, California

Benedict, Francis Gano, 1914 (9), Machiasport, Maine

Hulett, George Augustus, 1922 (5), 8 Greenholm, Princeton, New Jersey

Merrill, Elmer **D**rew, 1923 (7), Jamaica Plain, Massachusetts

Novy, Frederick **G**eorge, 1924 (10), 721 Forest Avenue, Ann Arbor, Michigan

Saunders, Frederick Albert, 1925 (3), South Hadley, Massachusetts

Schultz, Adolph Hans, 1939 (11), Anthropologisches Institut, Sempersteig 3, Zurich, Switzerland

Stratton, George **M**alcolm, 1928 (10), 1023 Life Sciences Building, University of California, Berkeley 4, California

Swanton, John Reed, 1932 (11), 22 George Street, Newton 58, Massachusetts

Tyzzer, Ernest **E**dward, 1942 (10), 175 Water Street, Wakefield, Massachusetts

Vandiver, Harry **S**hultz, 1934 (1), Box 7881, University of Texas, Austin 12, Texas

Whitney, Willis **R**odney, 1917 (5), General Electric Company Research Laboratory, P. O. Box 1088, Schenectady, New York

Number of Members Emeriti July 1, 1955: 12.

FOREIGN ASSOCIATES

The number in parentheses following the year of election indicates association within the sections of the National Academy of Sciences.

Adrian of **Cambridge,** Edgar Douglas, Baron, 1941 (9), Trinity College, Cambridge, England

Alexandroff, Paul A., 1947 (1), Mathematical Institute of the Academy of Sciences of the U.S.S.R., Bolshaya Kalushskaya 19, Moscow, U.S.S.R.

Bailey, Sir Edward, 1944 (6), 19 Greenhill Gardens, Edinburgh 10, Scotland

Bartlett, Sir Frederic Charles, 1947 (12), University of Cambridge, The Psychological Laboratory, Downing Place, Cambridge, England

Best, Charles **H**erbert, 1950 (9), Banting and Best Department of Medical Research, University of Toronto, Toronto, Canada

Bjerrum, Niels, 1952 (5), The Royal Veterinary and Agricultural College, Copenhagen, Denmark

Bohr, Niels, 1925 (3), Institute for Theoretical Physics, Blegdamsvej 15, Copenhagen, Denmark

Bordet, Jules, 1935 (10), Pasteur Institute, Rue du Remorquer, 28, Brussels, Belgium

Born, Max, 1955 (3), Marcard Strasse 4, Bad Pyrmont, Germany

Bragg, Sir William **Lawrence,** 1945 (3), Cavendish Laboratory, The University, Cambridge, England

de Broglie, Prince Louis, 1948 (3), 94 Rue Perronet, Neuilly-sur-Seine, France

Burnet, Sir Macfarlane, 1954 (10), The Walter and Eliza Hall Institute of Medical Research, Melbourne, Australia

Caso, Alfonso, 1943 (11), Av. Sudderman No. 312, Col. Chapultepec-Morales, Mexico 5, D. F.

Chapman, Sydney. 1946 (13), The Geophysical Institute, College, Alaska, U.S.A.

Dale, Sir Henry Hallett, 1940 (9), The Wellcome Trust, 24 Harley Street, London, W.1., England

Dirac, Paul Adrien **Maurice,** 1949 (1), Department of Mathematics, St. John's College, Cambridge, England

Eskola, Pentti Eelis, 1951 (6), Helsinki University, Snellmanink. 5, Helsinki, Finland

Fisher, Sir Ronald Aylmer, 1948 (8), Department of Genetics, University of Cambridge, 44 Storey's Way, Cambridge, England

von Frisch, Karl, 1951 (8), The Zoological Institute, University of Munich, Munich, Germany

Hadamard, Jacques, 1926 (1), 12 Rue Emile Faguet, Paris XIV, France

Helland-Hansen, Björn, 1947 (13), Chr. Michelsens Institutt for Videnscap, Bergen, Norway

Hill, Archibald Vivian, 1941 (9), 16 Bishopswood Road, Highgate, London, N.6, England

Houssay, Bernardo Alberto, 1940 (9), Viamonte 2790, Buenos Aires, Argentina

Jeffreys, Harold, 1945 (13), St. John's College, Cambridge, England

Jones, Sir Harold **Spencer,** 1943 (2), Royal Greenwich Observatory, Herstmonceux Castle, Hailsham, Sussex, England

Kapitza, Peter Leonidovich, 1946 (3), Institute for Physical Problems, Academy of Sciences of the U.S.S.R., Moscow, U.S.S.R.

Karrer, Paul, 1945 (5), University of Zurich, Zurich, Switzerland

Kluyver, Albert Jan, 1950 (7), Technical University, Delft, The Netherlands

Levi, Giuseppe, 1940 (8), Instituto di Anatomia Umana, Corso Massimo D'Azeglio, 52, Turin, Italy

Lim, Robert K. S., 1942 (9), The Miles-Ames Research Laboratory, Elkhart, Indiana

Lindblad, Bertil, 1955 (2), Stockholm Observatory, Saltsjöbaden, Sweden

Linderstrøm-Lang, Kaj Ulrik, 1947 (5), Chemical Division, Carlsberg Laboratory, Copenhagen, Denmark

Lwoff, André, 1955 (9), Institut Pasteur, Paris XV, France

Oort, Jan Hendrik, 1953 (1), Observatory of Leiden, Leiden, The Netherlands

Penfield, Wilder, 1953 (10), Montreal Neurological Institute, 3801 University Street, Montreal 2, Canada

Pieron, Henri, 1949 (12), Institute of Psychology, University of Paris, Paris, France

Portevin, Albert M. G. R., 1954 (4), 21, Boulevard de Beauséjour, Paris XVI, France

Reichstein, Tadeus, 1952 (5), Organisch-chemische Anstalt, St. Johanns-Ring 19, Basel, Switzerland

Renner, Otto, 1954 (7), The Botanical Institute, University of Munich, Munich, Germany

Robinson, Sir Robert, 1934 (5), Dyson Perrins Laboratory, South Parks Road, Oxford, England

Ruzicka, Leopold, 1944 (5), Department of Organic Chemistry, Institute of Technology, Zurich, Switzerland

Southwell, Sir Richard Vynne, 1943 (4), The Old House, Trumpington, Cambridge, England

Svedberg, The, 1945 (5), Fysikalisk-Kemiska Institutionen, Uppsala University, Uppsala, Sweden

Sverdrup, Harald Ulrik, (1945)* 1952 (13), Norsk Polarinstitutt, Observatoriegt 1, Oslo, Norway

Taylor, Sir Geoffrey Ingram, 1945 (1), Trinity College, Cambridge, England

Tiselius, Arne W. K., 1949 (14), Institute of Biochemistry, Uppsala University, Uppsala, Sweden

Todd, Sir Alexander Robertus, 1955 (5), University of Cambridge, University Chemical Laboratory, Pembroke Street, Cambridge, England

Vallée-Poussin, C. de la, 1929 (1), University of Louvain, Louvain, Belgium

Vening Meinesz, Felix Andries, 1939 (13), Potgieterlaan 5, Amersfoort, The Netherlands

Watson, D. M. S., 1938 (8), University College, Gower Street, London, W.C.1, England

Wieland, Heinrich, 1932 (5), Sophienstrasse 9, Munich 2 NW, Germany

Winge, Öjvind, 1949 (14), Department of Physiology, Carlsberg Laboratory, Copenhagen (Valby), Denmark

Yukawa, Hideki, 1949 (3), Yukawa Hall, Kyoto University, Kyoto, Japan

* Dr. Sverdrup was elected a member of the Academy in 1945. He resigned on April 2, 1951, on which date he had maintained residence in Norway for three years, thereby losing his American citizenship. He was elected a foreign associate on April 29, 1952.

Number of Foreign Associates July 1, 1955: 53

SECTIONS

(1) *Mathematics*—32 members

Smith, Paul A., *Chairman* (1958)
Ahlfors, L. V.
Albert, A. A.
Alexander, J. W.
Bell, E. T.
Bochner, S.
Brauer, R. D.
Chevalley, C.
Coble, A. B.
Courant, R.
Douglas, Jesse
Eisenhart, L. P.
Evans, G. C.
Gödel, Kurt
Hille, Einar
Jacobson, Nathan
Lefschetz, Solomon
Mac Lane, Saunders
McShane, E. J.
Montgomery, Deane
Moore, R. L.
Morse, Marston
Murnaghan, F. D.
Stone, M. H.
Thomas, T. Y.
Veblen, Oswald
von Neumann, John
Walsh, J. L.
Weyl, Hermann
Whitney, Hassler
Whyburn, G. T.
Zariski, O.

Foreign Associates

Alexandroff, P. A.
Dirac, P. A. M.
Hadamard, Jacques
Taylor, Sir Geoffrey
Vallée-Poussin, C. de la

(2) *Astronomy*—28 members

Struve, Otto, *Chairman* (1956)
Abbot, C. G.
Adams, W. S.
Babcock, H. D.
Babcock, H. W.
Bowen, I. S.
Brouwer, Dirk
Chandrasekhar, S.
Clemence, G. M.
Joy, A. H.
King, A. S.
Kuiper, G. P.
Mayall, N. U.
Menzel, D. H.
Merrill, P. W.
Mitchell, S. A.
Nicholson, S. B.
Ross, F. E.
Russell, H. N.
Seares, F. H.
Shapley, Harlow
Slipher, V. M.
Spitzer, Lyman, Jr.
Stebbins, Joel
Trumpler, R. J.
Whitford, A. E.
Wilson, Ralph E.
Wright, W. H.

Foreign Associates

Jones, Sir H. Spencer
Lindblad, Bertil
Oort, Jan Hendrik

(3) *Physics*—76 members

Van Vleck, J. H., *Chairman* (1957)
Allison, S. K.
Alvarez, L. W.
Anderson, C. D.
Bacher, R. F.
Bainbridge, K. T.
Bardeen, John
Beams, J. W.
Bethe, H. A.
Birge, R. T.
Bloch, Felix
Bradbury, N. E.
Breit, Gregory
Bridgman, P. W.
Brillouin, Leon
Brode, R. B.
Coblentz, W. W.
Compton, A. H.
Condon, E. U.
Coolidge, W. D.
Davis, Bergen
Davisson, C. J.
Dennison, D. M.
DuBridge, L. A.
DuMond, J. W. M.
Dunning, J. R.
Epstein, P. S.
Feynman, R. P.
Franck, James
Gamow, George
Goudsmit, S. A.
Herb, R. G.
Houston, W. V.
Hull, A. W.
Kemble, E. C.
Kent, R. H.
Kerst, D. W.
Lamb, W. E., Jr.
Land, E. H.
Lauritsen, C. C.
Lawrence, E. O.
Loomis, F. W.
Mason, Max
McMillan, E. M.
Meggers, W. F.
Morse, P. M.
Mulliken, R. S.

Nier, A. O. C.
O'Brien, Brian
Oppenheimer, J. R.
Panofsky, W. K. H.
Pegram, G. B.
Pierce, G. W.
Purcell, E. M.
Rabi, I. I.
Ramsey, N. F.
Robertson, H. P.
Rossi, Bruno
Schwinger, Julian
Segrè, E.
Seitz, Frederick
Serber, R.
Slater, J. C.
Stern, Otto
Stewart, G. W.
Street, J. C.
Teller, Edward
Tuve, M. A.
Uhlenbeck, G. E.
Webster, D. L.
Weisskopf, V. F.
Wheeler, J. A.
Wigner, E. P.
Wilson, Edwin B.
Wood, R. W.
Zachariasen, W. H.

Foreign Associates

Bohr, Niels
Born, Max
Bragg, Sir Lawrence
de Broglie, Prince Louis
Kapitza, P. L.
Yukawa, Hideki

(4) *Engineering*—40 members

Terman, F. E., *Chairman* (1956)
Adams, C. A.
Bain, E. C.
Briggs, L. J.
Buckley, O. E.
Bush, Vannevar
Chipman, John
Cochrane, E. L.
Curme, G. O., Jr.
Den Hartog, J. P.
Dryden, H. L.
Durand, W. F.
Fisk, J. B.
Fletcher, Harvey
Foote, P. D.
Gibbs, W. F.
Gilliland, E. R.
Greenewalt, C. H.
Hoover, Herbert
Hunsaker, J. C.
Jeffries, Zay
Kelly, M. J.
Kettering, C. F.
Lewis, W. K.
Loomis, A. L.
Merica, P. D.
Murphree, E. V.
Pierce, J. R.
Raymond, A. E.
Savage, J. L.
Shockley, W.
Slepian, Joseph
Soderberg, C. R.
Stratton, J. A.
Suits, C. G.
Thomas, C. A.
Timoshenko, Stephen
von Kármán, T.
Wilson, Robert E.
Zworykin, V. K.

Foreign Associates

Portevin, Albert M. G. R.
Southwell, Sir Richard V.

(5) *Chemistry*—75 members

Kistiakowsky, G. B., *Chairman* (1956)
Adams, Roger
Badger, R. M.
Bartlett, P. D.
Bolton, E. K.
Brode, W. R.
Calvin, Melvin
Conant, J. B.
Cope, A. C.
Craig, L. C.
Daniels, Farrington
Debye, Peter
Edsall, J. T.
Elderfield, R. C.
Emmett, P. H.
Eyring, Henry
Fieser, L. F.
Flory, P. J.
Folkers, Karl
Fuoss, R. M.
Fuson, R. C.
Giauque, W. F.
Gilman, Henry
Hammett, L. P.
Harned, H. S.
Hendricks, S. B.
Hildebrand, J. H.
Hirschfelder, J. A.
Jacobs, W. A.
Johnson, J. R.
Johnson, W. S.
Keyes, F. G.
Kharasch, M. S.
Kimball, G. E.
Kirkwood, J. G.
Kraus, C. A.
La Mer, V. K.
Langmuir, Irving
Leonard, N. J.
Libby, W. F.
Lind, S. C.

Longsworth, L. G.
MacInnes, D. A.
Marvel, C. S.
Mayer, J. E.
McElvain, S. M.
Mees, C. E. K.
Niemann, C. G.
Noyes, W. A., Jr.
Onsager, Lars
Pauling, Linus
Pitzer, K. S.
Rodebush, W. H.
Rossini, F. D.
Scatchard, George
Schlesinger, H. I.
Seaborg, G. T.
Shedlovsky, Theodore
Small, L. F.
Smith, L. I.
Smyth, C. P.
Spedding, F. H.
Tishler, Max
Urey, H. C.
Westheimer, F. H.
Williams, J. W.
Williams, Robert R.
Williams, Roger J.
Wilson, E. Bright, Jr.
Winstein, Saul
Wolfrom, M. L.
Woodward, R. B.
Wyckoff, R. W.
Yost, D. M.
Young, W. G.

Foreign Associates

Bjerrum, Niels
Karrer, Paul
Linderstrøm-Lang, K. U.
Reichstein, Tadeus
Robinson, Sir Robert
Ruzicka, Leopold
Svedberg, The
Todd, Sir Alexander
Wieland, Heinrich

(6) *Geology*—37 members

Gilluly, James, *Chairman* (1957)
Allen, E. T.
Berkey, C. P.
Birch, Francis
Blackwelder, Eliot
Bowen, N. L.
Bradley, W. H.
Bramlette, M. N.
Bucher, W. H.
Buddington, A. F.
Buerger, M. J.
Chaney, R. W.
Cloos, Ernst
Daly, R. A.
Day, A. L.
DeGolyer, E. L.
Dunbar, C. O.
Gregory, W. K.
Hess, H. H.
Hewett, D. F.
Hubbert, M. King
Kelley, W. P.
Knopf, Adolph
Larsen, E. S., Jr.
Leith, C. K.
Longwell, C. R.
Lovering, T. S.
Mead, W. J.
Mendenhall, W. C.
Nolan, T. B.
Reeside, J. B., Jr.
Rubey, W. W.
Ruedemann, Rudolf
Schairer, J. F.
Simpson, G. G.
Williams, Howel
Woodring, W. P.

Foreign Associates

Bailey, Sir Edward
Escola, Pentti

(7) *Botany*—42 members

Raper, K. B., *Chairman* (1956)
Anderson, Edgar
Bailey, I. W.
Barker, H. A.
Beadle, G. W.
Blinks, L. R.
Bonner, James
Brink, R. A.
Burkholder, P. R.
Chandler, W. H.
Clausen, R. E.
Cleland, R. E.
Couch, J. N.
Delbrück, Max
Dodge, B. O.
Duggar, B. M.
Emerson, Robert
Fred, E. B.
Goddard, D. R.
Horsfall, J. G.
Jones, D. F.
Kaufmann, B. P.
Kunkel, L. O.
Mangelsdorf, P. C.
McClintock, Barbara
Osterhout, W. J. V.
Rhoades, M. M.
Riker, A. J.
Robbins, W. J.
Sax, Karl
Sinnott, E. W.
Smith, Gilbert M.
Stakman, E. C.
Stebbins, G. L.
Thimann, K. V.

Thom, Charles
Van Niel, C. B.
Waksman, S. A.
Walker, J. C.
Went, F. W.
Wetmore, R. H.
Wilson, P. W.

Foreign Associates

Renner, Otto
Kluyver, A. J.

(8) *Zoölogy and Anatomy*—50 members

Nicholas, J. S., *Chairman* (1958)
Bartelmez, G. W.
Bigelow, H. B.
Bloom, William
Burns, R. K.
Castle, W. E.
Cleveland, L. R.
Corner, G. W.
Danforth, C. H.
Demerec, Milislav
Detwiler, S. R.
Dobzhansky, Th.
Dunn, L. C.
Goldschmidt, R. B.
Hamburger, Viktor
Harrison, R. G.
Hartman, C. G.
Harvey, E. N.
Herrick, C. J.
Hisaw, F. L.
Holtfreter, J.
Hubbs, C. L.
Hutchison, G. E.
Irwin, M. R.
Jacobs, M. H.
Kellogg, Remington
Lewis, W. H.
Mayr, Ernst
Metz, C. W.
Mirsky, A. E.
Moore, C. R.
Muller, H. J.
Painter, T. S.
Patterson, J. T.
Petrunkevitch, Alexander
Riddle, Oscar
Romer, A. S.
Schmitt, F. O.
Schrader, Franz
Smith, Philip E.
Sonneborn, T. M.
Stern, Curt
Sturtevant, A. H.
Taliaferro, W. H.
Twitty, V. C.
Weiss, Paul
Wetmore, Alexander
Willier, B. H.
Wislocki, G. B.
Wright, Sewall

Foreign Associates

Fisher, Sir Ronald
von Frisch, Karl
Levi, Giuseppe
Watson, D. M. S.

(9) *Physiology*—27 members

Fenn, W. O., *Chairman* (1957)
Bard, Philip
Bronk, Detlev W.
Carlson, A. J.
Davis, Hallowell
DuBois, E. F.
Erlanger, Joseph
Evans, H. M.
Forbes, Alexander
Gasser, H. S.
Gerard, R. W.
Hartline, H. K.
Hastings, A. B.
Landis, E. M.
Lloyd, D. P. C.
Loeb, R. F.
Long, C. N. H.
Lorente de Nó, R.
Magoun, H. W.
Mann, Frank
Marshall, E. K., Jr.
Meek, W. J.
Peters, J. P.
Richards, A. N.
Schmidt, C. F.
Smith, Homer W.
Wiggers, C. J.

Foreign Associates

Lord Adrian
Best, C. H.
Dale, Sir Henry
Hill, A. V.
Houssay, B. A.
Lim, Robert K. S.
Lwoff, André

(10) *Pathology and Bacteriology*—37 members

Cannon, P. R., *Chairman* (1957)
Albright, Fuller
Armstrong, Charles
Bailey, P.
Blalock, Alfred
Castle, W. B.
Coggeshall, L. T.
Cole, Rufus
Dalldorf, Gilbert
Dochez, A. R.
Dragstedt, L. R.
Dubos, R. J.
Enders, J. F.
Francis, Thomas, Jr.
Gamble, J. L.
Goodpasture, E. W.
Graham, E. A.
Heidelberger, Michael
Horsfall, F. L., Jr.
Huggins, C. B.
Little, C. C.
Loeb, Leo
Long, E. R.
MacLeod, C. M.
Maxcy, K. F.
McMaster, P. D.
Meyer, K. F.
Opie, E. L.
Paul, J. R.
Rich, A. R.
Rivers, T. M.
Robertson, O. H.
Rous, Peyton
Sabin, Albert B.
Shope, R. E.
Tillett, W. S.
Whipple, G. H.

Foreign Associates

Bordet, Jules
Burnet, Sir Macfarlane
Penfield, Wilder

(11) *Anthropology*—10 members

Shapiro, H. L., *Chairman* (1957)
Albright, W. F.
Coon, C. S.
Kidder, A. V.
Kluckhohn, C. K. M.
Kroeber, A. L.
Lothrop, S. K.
Lowie, R. H.
Spier, Leslie
Steward, J. H.

Foreign Associate

Caso, Alfonso

(12) *Psychology*—24 members

Graham, C. H., *Chairman* (1956)
Beach, F. A.
Boring, E. G.
Carmichael, Leonard
Gesell, Arnold
Guilford, J. P.
Harlow, H. F.
Hilgard, E. R.
Köhler, Wolfgang
Lashley, K. S.
Lindsley, D. B.
Miles, W. R.
Nissen, H. W.
Pillsbury, W. B.
Richter, C. P.
Skinner, B. F.
Spence, K. W.
Stevens, S. S.
Terman, L. M.
Thurstone, L. L.
Tolman, E. C.
Wever, E. G.
Woodworth, R. S.
Yerkes, R. M.

Foreign Associates

Bartlett, Sir Frederic
Pieron, Henri

(13) *Geophysics*—24 members

Berkner, L. V., *Chairman* (1957)
Adams, L. H.
Benioff, Hugo
Bjerknes, J.
Brown, Harrison
Byerly, Perry
Byers, H. R.
Eckart, Carl
Ewing, Maurice
Fleming, J. A.
Griggs, D. T.
Gunn, Ross
Gutenberg, Beno
Iselin, C. O'D.
Lambert, W. D.
Macelwane, J. B.
Pekeris, C. L.

Piggot, C. S.
Reichelderfer, F. W.
Rossby, C.-G.
Slichter, L. B.
Thompson, T. G.
Vestine, E. H.
Wulf, O. R.

Foreign Associates

Chapman, Sydney
Helland-Hansen, Björn
Jeffreys, H.
Sverdrup, H. U.
Vening Meinesz, F. A.

(14) *Biochemistry*—40 members

Stanley, W. M., *Chairman* (1958)
Anderson, R. J.
Ball, E. G.
Balls, A. K.
Carter, H. E.
Chance, Britton
Clark, W. M.
Clarke, H. T.
Cori, Carl F.
Cori, Gerty
Doisy, E. A.
du Vigneaud, V.
Elvehjem, C. A.
Fischer, H. O. L.
Fruton, J. S.
Kendall, E. C.
King, C. G.
Link, K. P.
Lipmann, Fritz
Maynard, L. A.
McCollum, E. V.
Northrop, J. H.
Oncley, J. L.
Rittenberg, David
Rose, W. C.
Shaffer, P. A.
Sherman, H. C.
Snell, E. E.
Stadie, W. C.
Sumner, J. B.
Tatum, E. L.
Van Slyke, D. D.
Vickery, H. B.
Wald, George
Werkman, C. H.
Williams, R. C.
Wilson, D. W.
Wintersteiner, O.
Wood, H. G.
Woolley, D. W.

Foreign Associates

Tiselius, Arne W. K.
Winge, Öjvind

GEOGRAPHICAL LISTING of ACADEMY MEMBERS

Arizona

Slipher, V. M.

California

Adams, W. S.
Alvarez, L. W.
Anderson, C. D.
Anderson, J. A.
Babcock, H. D.
Babcock, H. W.
Bacher, R. F.
Badger, R. M.
Barker, H. A.
Beadle, G. W.
Bell, E. T.
Benioff, Hugo
Birge, R. T.
Bjerknes, J.
Blackwelder, Eliot
Blinks, L. R.
Bonner, James
Bowen, I. S.
Bramlette, M. N.
Brode, R. B.
Brown, Harrison
Byerly, Perry
Calvin, Melvin
Castle, W. E.
Chandler, W. H.
Chaney, R. W.
Clausen, R. E.
Coble, A. B.
Condon, E. U.
Danforth, C. H.
Delbrück, Max
DuBridge, L. A.
DuMond, J. W. M.
Eckart, Carl
Epstein, P. S.
Evans, G. C.
Evans, H. M.
Feynman, R. P.
Fischer, H. O. L.
Giauque, W. F.
Goldschmidt, R. B.
Griggs, D. T.
Guilford, J. P.
Gutenberg, Beno
Hewett, D. F.
Hildebrand, J. H.
Hilgard, E. R.
Hubbs, C. L.
Joy, A. H.
Kelley, W. P.
King, A. S.
Knopf, Adolph
Kroeber, A. L.

Lamb, W. E., Jr.
Lauritsen, C. C.
Lawrence, E. O.
Lindsley, D. B.
Longwell, C. R.
Lowie, R. H.
Magoun, H. W.
Mason, Max
Mayall, N. U.
McMillan, E. M.
Merrill, P. W.
Meyer, K. F.
Nicholson, S. B.
Niemann, Carl
Northrop, J. H.
Panofsky, W. K. H.
Pauling, Linus
Pitzer, K. S.
Raymond, A. E.
Robertson, O. H.
Ross, F. E.
Seaborg, G. T.
Seares, F. H.
Segrè, Emilio
Slichter, L. B.
Smith, G. M.
Spier, Leslie
Stanley, W. M.
Stebbins, G. L.
Stebbins, Joel
Stern, Curt
Stern, Otto
Stratton, G. M.
Struve, Otto
Sturtevant, A. H.
Tatum, E. L.
Teller, Edward
Terman, F. E.
Terman, L. M.
Timoshenko, S. P.
Tolman, E. C.
Trumpler, R. J.
Twitty, V. C.
Van Niel, C. B.
von Kármán, T.
Webster, D. L.
Went, F. W.
Williams, Howel
Williams, R. C.
Wilson, Ralph E.
Winstein, Saul
Wright, W. H.
Wulf, O. R.
Yost, D. M.
Young, W. G.

Colorado

Gilluly, James
Lovering, T. S.
Savage, J. L.

Connecticut

Anderson, R. J.
Beach, F. A.
Breit, G.
Brouwer, Dirk
Dunbar, C. O.
Fruton, J. S.
Fuoss, R. M.
Gesell, Arnold
Harned, H. S.
Harrison, R. G.
Hille, Einar
Horsfall, J. G.
Hutchinson, G. E.
Jacobson, Nathan
Jones, D. F.
Kirkwood, J. G.
Lambert, W. D.
Long, C. N. H.
Nicholas, J. S.
Onsager, Lars
Paul, J. R.
Peters, J. P.
Petrunkevitch, A.
Sinnott, E. W.
Vickery, H. B.
Yerkes, R. M.

Delaware

Bolton, E. K.
Greenewalt, C. H.

District of Columbia

Abbot, C. G.
Adams, L. H.
Bowen, Norman L.
Bradley, W. H.
Briggs, L. J.
Brode, W. R.
Bush, V.
Carmichael, Leonard
Clemence, G. M.
Coblentz, W. W.
Dryden, H. L.
Gamow, George
Goodpasture, E. W.
Gunn, Ross
Kellogg, Remington
Larsen, E. S., Jr.
Leith, C. K.
Libby, W. F.
Meggers, W. F.
Nolan, T. B.
Piggot, C. S.
Reichelderfer, F. W.
Rubey, W. W.
Schairer, J. F.
Tuve, M. A.
Vestine, E. H.
von Neumann, John
Wetmore, Alexander
Woodring, W. P.

Florida

Lashley, K. S.
Nissen, H. W.
Riddle, Oscar

Georgia

Burkholder, P. R.

Illinois

Adams, Roger
Albert, A. A.
Allison, S. K.
Bailey, P.
Bardeen, John
Bloom, William
Byers, H. R.
Cannon, Paul
Carlson, A. J.
Carter, H. E.
Coggeshall, L. T.
Dragstedt, L. R.
Emerson, Robert
Franck, James
Fuson, R. C.
Gerard, R. W.
Huggins, Charles
Kerst, D. W.
Kharasch, M. S.
Leonard, N. J.
Loomis, F. W.
Mac Lane, Saunders

Marvel, C. S.
Mayer, J. E.
Moore, C. R.
Mulliken, R. S.
Rhoades, M. M.
Rodebush, W. H.
Rose, W. C.
Schlesinger, H. I.
Seitz, Frederick
Steward, J. H.
Stone, M. H.
Taliaferro, W. H.
Urey, H. C.
Wilson, Robert E.
Zachariasen, W. H.

Indiana

Balls, A. K.
Cleland, R. E.
*Lim, R. K. S.
Muller, H. J.
Sonneborn, T. M.
Thomas, T. Y.

Iowa

Gilman, Henry
Spedding, F. H.
Spence, K. W.
Stewart, G. W.
Werkman, C. H.

Maine

Benedict, F. G.
Little, C. C.

Maryland

Albright, W. F.
Armstrong, Charles
Bard, Philip
Bartelmez, G. W.
Blalock, Alfred
Burns, R. K.
Clark, W. M.
Cloos, Ernst
Corner, G. W.
Day, A. L.
Emmett, P. H.
Hendricks, S. B.

* Foreign Associate.

Kent, R. H.
Marshall, E. K., Jr.
Maxcy, K. F.
McCollum, E. V.
Mendenhall, W. C.
Murnaghan, F. D.
Reeside, J. B., Jr.
Rich, A. R.
Richter, C. P.
Small, L. F.
Willier, B. H.
Wood, R. W.
Wyckoff, R. W. G.

Massachusetts

Ahlfors, L. V.
Albright, Fuller
Allen, E. T.
Bailey, I. W.
Bainbridge, K. T.
Ball, E. G.
Bartlett, P. D.
Bigelow, H. B.
Birch, Francis
Boring, E. G.
Brauer, R.
Bridgman, P. W.
Buerger, M. J.
Castle, W. B.
Chipman, John
Cleveland, L. R.
Cochrane, E. L.
Cope, A. C.
Daly, R. A.
Den Hartog, J. P.
Edsall, J. T.
Enders, J. F.
Fieser, L. F.
Forbes, Alexander
Gamble, J. L.
Gilliland, E. R.
Hastings, A. B.
Hisaw, F. L.
Hunsaker, J. C.
Iselin, C. O'D.
Jeffries, Zay
Kemble, E. C.
Keyes, F. G.
Kidder, A. V.
Kistiakowsky, G. B.
Kluckhohn, Clyde

Land, E. H.
Landis, E. M.
Lewis, W. K.
Lipmann, Fritz
Lothrop, S. K.
Mangelsdorf, P. C.
Mayr, Ernst
Mead, W. J.
Menzel, D. H.
Merrill, E. D.
Morse, P. M.
O'Brien, Brian
Oncley, J. L.
Pierce, G. W.
Purcell, E. M.
Ramsey, Norman
Romer, A. S.
Rossi, B. B.
Saunders, F. A.
Sax, Karl
Scatchard, George
Schmitt, F. O.
Schwinger, Julian
Skinner, B. F.
Slater, J. C.
Smith, P. E.
Soderberg, C. R.
Stevens, S. S.
Stratton, J. A.
Street, J. C.
Swanton, J. R.
Thimann, K. V.
Tyzzer, E. E.
Van Vleck, J. H.
Wald, George
Walsh, J. L.
Weisskopf, V. F.
Westheimer, F. H.
Wetmore, R. H.
Wilson, E. Bright, Jr.
Wilson, Edwin B.
Wislocki, G. B.
Woodward, R. B.
Zariski, Oscar

Michigan

Dennison, D. M.
Elderfield, R. C.
Francis, Thomas, Jr.
Herrick, C. J.
Kettering, C. F.

Novy, F. G.
Pillsbury, W. B.
Uhlenbeck, G. E.

Minnesota

Mann, F. C.
Nier, A. O. C.
Smith, L. I.
Stakman, E. C.

Missouri

Anderson, Edgar
Compton, A. H.
Cori, C. F.
Cori, G. T.
Davis, Hallowell
Doisy, E. A.
Erlanger, Joseph
Graham, E. A.
Hamburger, Viktor
Loeb, Leo
Macelwane, J. B.
Shaffer, P. A.
Thomas, C. A.

New Hampshire

Shapley, Harlow

New Jersey

Alexander, J. W.
Bochner, S.
Buckley, O. E.
Buddington, A. F.
Eisenhart, L. P.
Fisk, J. B.
Folkers, Karl
Gödel, Kurt
Hartman, C. G.
Harvey, E. N.
Heidelberger, M.
Hess, H. H.
Hulett, G. A.
Kendall, E. C.
Köhler, Wolfgang
Lefschetz, Solomon
Montgomery, Deane
Morse, Marston
Oppenheimer, J. R.
Pierce, J. R.
Russell, H. N.
Shockley, W.
Smyth, C. P.
Spitzer, Lyman, Jr.
Tishler, Max
Veblen, Oswald
Waksman, S. A.
Wever, E. G.
Weyl, Hermann
Wheeler, J. A.
Whitney, Hassler
Wigner, E. P.
Williams, R. R.
Wintersteiner, Oskar
Zworykin, V. K.

New Mexico

Bradbury, N. E.

New York

Berkey, C. P.
Berkner, L. V.
Bethe, H. A.
Brillouin, Leon
Bronk, D. W.
Bucher, W. H.
Chevalley, C.
Clarke, H. T.
Cole, Rufus
Coolidge, W. D.
Courant, Richard
Craig, L. C.
Curme, G. O., Jr.
Dalldorf, Gilbert
Davis, Bergen
Debye, Peter
Demerec, Milislav
Detwiler, S. R.
Dobzhansky, Th.
Dochez, A. R.
Dodge, B. O.
Douglas, Jesse
DuBois, E. F.
Dubos, R. J.
Duggar, B. M.
Dunn, L. C.
Dunning, J. R.
Durand, W. F.
du Vigneaud, Vincent
Ewing, Maurice
Fenn, W. O.
Flory, P. J.
Gasser, H. S.
Gibbs, W. F.
Goudsmit, S. A.
Graham, C. H.
Gregory, W. K.
Hammett, L. P.
Hartline, H. K.
Holtfreter, J.
Hoover, Herbert
Horsfall, F. L., Jr.
Hull, A. W.
Jacobs, W. A.
Johnson, J. R.
Kaufmann, B. P.
Kelly, M. J.
Kimball, G. E.
King, C. G.
Kunkel, L. O.
LaMer, V. K.
Langmuir, Irving
Lloyd. D. P. C.
Loeb, R. F.
Longsworth, L. G.
Loomis, A. L.
Lorente de Nó, R.
MacInnes, D. A.
MacLeod, C. M.
Maynard, L. A.
McClintock, Barbara
McMaster, P. D.
Mees, C. E. K.
Merica, P. D.
Mirsky, A. E.
Murphree, E. V.
Noyes, W. A., Jr.
Opie, E. L.
Osterhout, W. J. V.
Pegram, G. B.
Rabi, I. I.
Rittenberg, David
Rivers, T. M.
Robbins, W. J.
Rous, Peyton
Ruedemann, Rudolf
Schrader, Franz
Serber, Robert
Shapiro, H. L.
Shedlovsky, Theodore
Sherman, H. C.
Shope, R. E.
Simpson, G. G.
Smith, H. W.
Smith, P. A.
Suits, C. G.

Sumner, J. B.
Thom, Charles
Tillett, W. S.
Van Slyke, D. D.
Weiss, Paul
Whipple, G. H.
Whitney, W. R.
Woodworth, R. S.
Woolley, D. W.

North Carolina

Couch, J. N.
Thurstone, L. L.

Ohio

Sabin, A. B.
Wiggers, C. J.
Wolfrom, M. L.
Wood, H. G.

Pennsylvania

Adams, C. A.
Bain, E. C.
Chance, Britton
Coon, C. S.
Foote, P. D.
Goddard, D. R.
Jacobs, M. H.
Lewis, W. H.
Long, E. R.
Metz, C. W.
Richards, A. N.
Rossini, F. D.
Schmidt, C. F.
Slepian, Joseph
Stadie, W. C.
Wilson, D. W.

Rhode Island

Kraus, C. A.

Tennessee

Fleming, J. A.
Lind, S. C.

Texas

DeGolyer, E.
Houston, W. V.
Hubbert, M. K.
Moore, R. L.
Painter, T. S.
Patterson, J. T.
Snell, E. E.
Vandiver, H. S.
Williams, R. J.

Utah

Eyring, Henry
Fletcher, Harvey

Virginia

Beams, J. W.
Davisson, C. J.
McShane, E. J.
Mitchell, S. A.
Whyburn, G. T.

Washington

Thompson, T. G.

Wisconsin

Brink, R. A.
Chandrasekhar, S.
Daniels, Farrington
Elvehjem, C. A.
Fred, E. B.
Harlow, H. F.
Herb, R. G.
Hirschfelder, J. O.
Irwin, M. R.
Johnson, W. S.
Kuiper, G. P.
Link, K. P.
McElvain, S. M.
Meek, W. J.
Raper, K. B.
Riker, A. J.
Walker, J. C.
Whitford, A. E.
Williams, J. W.
Wilson, P. W.
Wright, Sewall

Members Abroad

France—

Robertson, H. P.

Germany—

Conant, J. B.

Israel—

Pekeris, C. L.

Sweden—

Rossby, C.-G.

Switzerland—

Bloch, Felix
Schultz, A. H.

Turkey

Miles, W. R.

INFORMATION TO CONTRIBUTORS

THE PROCEEDINGS is the official organ of the NATIONAL ACADEMY OF SCIENCES and of the NATIONAL RESEARCH COUNCIL for the publication of brief accounts of important current researches of members of the ACADEMY and of the COUNCIL and of other American investigators. The PROCEEDINGS will aim especially to secure prompt publication of original announcements of discoveries and wide circulation of the results of American research among investigators in other countries and in all branches of science.

ARTICLES should be brief. The viewpoint should be comprehensive in giving the relation of the paper to previous publications of the author or of others and in exhibiting, where practicable, the significance of the work for other branches of science. Elaborate technical details of the work and long tables of data should be avoided, but authors should be precise in making clear the new results and should give some record of the methods and data upon which they are based.

MANUSCRIPTS should be prepared with a current number of the PROCEEDINGS as a model in matters of form, and should be typewritten in duplicate with double spacing, the author retaining one copy. Illustrations should be confined to text figures of simple character, though more elaborate illustration may be allowed in special instances to authors willing to pay for their preparation and insertion. Particular attention should be given to arranging tabular matter in a simple and concise manner.

MANUSCRIPTS will be accepted only from members of the Academy or from chairmen of the divisions of the National Research Council who will assume responsibility for the propriety and scientific standards of the paper and should be addressed to *Proceedings of the* NATIONAL ACADEMY OF SCIENCES, Attention Miss Mary D. Alexander, University of Chicago Press, 5750 Ellis Avenue, Chicago 37, Illinois. Articles of which a member of the Academy is author or co-author may extend to 10 printed pages, those by non-members to 3 printed pages; space above these limits will be billed to the author at a rate based on the actual composition costs of the article. Accompanying the script should be a statement by the author or his representative that he assumes responsibility for the charges if incurred.

GALLEY PROOF will be sent, with the understanding that charges for author's corrections shall be billed to him. Author should therefore make final revisions on the typewritten manuscripts. Page proofs will not be sent.

REPRINTS should be ordered when proof is returned, on the reprint order form which will be inclosed with the proof. The order blank gives a tabular estimation of the cost of reprints; a copy will be supplied at any time upon request.

CONTENTS

Page

ASTRONOMY.—SOME CRITICISMS OF "ON THE ORIGIN OF THE LUNAR SURFACE FEATURES" By G. P. KUIPER . . . . . *By Harold C. Urey* 423

BIOCHEMISTRY.—EFFECT OF ADENINE MONONUCLEOTIDES ON NEURAL TUBE FORMATION OF FROG EMBRYO . . . . . *By Elisabeth Ambellan* 428

BIOCHEMISTRY.—A RAPID METHOD FOR VIABLE CELL TITRATION AND CLONE PRODUCTION WITH HELA CELLS IN TISSUE CULTURE: THE USE OF X-IRRADIATED CELLS TO SUPPLY CONDITIONING FACTORS . . . . . *By Theodore T. Puck and Philip I. Marcus* 432

BIOCHEMISTRY.—HINDERED CIS ISOMERS OF VITAMIN A AND RETINENE: THE STRUCTURE OF THE NEO-B ISOMER . . . . *By George Wald, Paul K. Brown, and Ruth Hubbard* 438

ENGINEERING.—ISOTOPE SEPARATION BY IONIC EXPANSION IN A MAGNETIC FIELD . . . . . *By Joseph Slepian* 451

GEOLOGY.—GRAVITY ANOMALIES AT CONTINENTAL MARGINS . . . . . *By J. Lamar Worzel and G. Lynn Shurbet* 458

GEOPHYSICS.—THE SEISMIC SURFACE PULSE . . . . . *By C. L. Pekeris* 469

MATHEMATICS.—ON INVOLUTORIAL ALGEBRAS . . . . . *By A. A. Albert* 480

MATHEMATICS.—FUNCTIONAL EQUATIONS IN THE THEORY OF DYNAMIC PROGRAMMING. II. NONLINEAR DIFFERENTIAL EQUATIONS . . . . . *By Richard Bellman* 482

MATHEMATICS.—STATIONARY SPACE-TIME IN GENERAL RELATIVITY . . *By S. Bochner* 485

MATHEMATICS.—THE COHOMOLOGY RING OF G/T . . . . *By R. Bott and H. Samelson* 490

MATHEMATICS.—GEOMETRIC ASPECTS OF CURRENTS AND DISTRIBUTIONS . . . . . *By James Eells, Jr.* 493

MATHEMATICS.—GENERAL FACTORS OF AUTOMORPHY . . . . . *By R. C. Gunning* 496

MATHEMATICS.—DIGITAL COMPUTATIONAL METHODS IN SYMBOLIC LOGIC, WITH EXAMPLES IN BIOCHEMISTRY . . . . . *By Robert S. Ledley* 498

MATHEMATICS.—THE BEHAVIOR OF A COMPLEX FUNCTION AT INFINITY . *By Karl Menger* 512

MATHEMATICS.—ON THE GENERATION OF THE IRREDUCIBLE REPRESENTATIONS OF THE SYMMETRIC GROUP . . . . . *By Francis D. Murnaghan* 514

MATHEMATICS.—ON THE ANALYSIS OF THE KRONECKER PRODUCT OF IRREDUCIBLE REPRESENTATIONS OF S_n . . . . . *By Francis D. Murnaghan* 515

MATHEMATICS.—AN APPLICATION OF ALGEBRAIC TOPOLOGY TO NUMERICAL ANALYSIS: ON THE EXISTENCE OF A SOLUTION TO THE NETWORK PROBLEM . . . . *By J. P. Roth* 518

ZOÖLOGY.—THE ROLES OF THE OPTIC VESICLE AND OTHER HEAD TISSUES IN LENS INDUCTION . . . . . *By Antone G. Jacobson* 522

N. A. S. ORGANIZATION 527